Introduction to Knot Concordance

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CHAPTER 1

Introduction

Loosely speaking, the goal of this book is to provide a usable while consise discussion of some of the fundamental tools on knot concordance, in order to give a solid base from which to work. We begin this introduction with a short historical motivation for the study of concordance.

Oriented knots K and J in S^3 are concordant if the connected sum $K \# -J$, bounds a smoothly embedded disk in the four-ball B^4; here $-J$ denotes the mirror image of J with string orientation reversed. If a knot itself bounds such an embedded disk in B^4, the knot is called *slice*. An equivalent definition is roughly stated as follows: K and J are concordant if there is an embedding $f\colon A \to S^3 \times [0,1]$, where $A \cong S^1 \times [0,1]$, $f(S^1 \times \{0\}) \cong K$ and $f(S^1 \times \{1\}) \cong -J$. A reader should then think of these as natural extensions of 3-dimensional properties of knots in the 4'th dimension. While the trivial knot is the only knot in S^3 which bounds an embedded disk in S^3, concordance regards any knot that bounds a disk in B^4 as trivial. Similarly, the notion of concordance as cobounding an embedded annulus is inspired by considering the trace of isotopy as a map from $S^1 \times [0,1]$ to $S^3 \times [0,1]$, and the allowing for some critial points.

The differences between studying knots in S^3 up to isotopy and up to concordance are significant. For instance, consider the connected sum. This is an associative and commutative binary operation on the set of isotopy classes of knots. An immediate consequence of additivity of knot genus is that this operation does not induce a group structure on the set of knots. Switching to concordance remedies this: the set of concordance classes of knots forms a group under the connected sum operation. Making this precise is not difficult, given basic results in geometric topology. The goal of this book is to study the knot concordance group from algebraic and topological perspectives and to present many of the main results related to its structure with a focus on finite order classes in the algebraic concordance group and the concordance order of knots that represent such classes.

The proof that the concordance group, $\mathcal{C}$, is a well-defined countable abelian group is elementary, contained in a short paper by Fox and Milnor in [?]. Viewed purely as an abstract group, all that is known about $\mathcal{C}$ is that it splits off an infinitely generated free summand and an infinite summand consisting of two-torsion. From a topological perspective, there is more of interest to study. In particular, there are surjections of $\mathcal{C}$ to higher dimensional concordance groups, which were completely classified by Levine [?], and also a surjection to the topological (as opposed to smooth) concordance group. It is known that both kernels have infinite free summands and infinite two-torsion.

The early study of concordance had two distinct motivations. The first came from discoveries related to two-dimensional knots and links embedded in four-dimensional space. Artin [?] was the first to observe that nontrivial 1-dimensional knots in the 3-sphere could bound embedded disks in the four-ball. (More precisely, he showed that there are knots in $\mathbb{R}^3$ that are cross-sections of knotted two-spheres embedded in $\mathbb{R}^4$.) More surprising examples were discovered later, including embeddings of S^2 into $\mathbb{R}^4$ which are unknotted, but have non-trivial knots as cross-sections. There are other such unknotted two spheres with cross-sections that are nontrivially linked knots. And there are also pairs of spheres embedded in $\mathbb{R}^4$ which form the unlink but which have cross-sections that are nontrivial links of two components. Concordance provided a valuable perspective on such examples.

From a modern viewpoint, a motivation that arose from algebraic geometry is more important. A homogenous degree d polynomial in three complex variables defines an algebraic curve in projective space, $\mathbb{C}\mathbf{P}^2$. Generically, this curve is smooth, of genus $(d-1)(d-2)/2$, and it represents $d \in H_2(\mathbb{C}\mathbf{P}^2) \cong \mathbb{Z}$. The Thom Conjecture, which was ultimately proved by Kronheimer and Mrowka [?], states that $(d-1)(d-2)/2$ is the least genus among all *smooth* surfaces representing $d \in H_2(\mathbb{C}\mathbf{P}^2) \cong \mathbb{Z}$.

One approach to finding counterexamples to the Thom Conjecture (an approach that actually works in the topological, locally flat category) is the following. Begin with a singular curve having isolated singularities that are cones on knots, K_i. Cut out those cones and replace them with smooth surfaces in $\mathbf{B}^4$ of low genus having the K_i as boundary. As an example, consider the curve defined by $x^2z + y^3 = 0$. This is topologically a sphere which is smooth except at one singular point, a cone on the trefoil knot. If the trefoil knot were trivial in the concordance group, it would bound a smooth disk in $\mathbf{B}^4$, which could replace the cone to yield a smooth sphere, which is a genus zero surface, representing $3 \in H_1(\mathbb{C}\mathbf{P}^2)$. The Thom Conjecture states that the least genus representative is of genus one; thus, if the trefoil were slice, we would have a counterexample. This approach leads immediately to the following definition: the four-genus of a knot K, $g_4(K)$, is the minimum genus of a smooth surface in the four-ball bounded by K. It is a simple observation that the four-genus induces a function $g_4\colon \mathcal{C} \to \mathbb{Z}$. This function is of continuing research interest. In this book we will present some of the fundamental results concerning the four-genus.

Beyond these two original motivations for studying concordance, there is an overriding interest. Each advance in low-dimensional geometric topology has been accompanied by advances in knot concordance. To mention a few instances of this: surgery theory led to Levine's classification of knot concordance groups; the $G-$signature theorem provided the starting point for Casson-Gordon's work; Donaldson's introduction of gauge theory offered entirely new techniques to study concordance, for instance providing examples of topologically slice knots that are not smoothly slice; techniques from Heegaard-Floer similarly led to ground-breaking advances in our understanding of the concordance group. The knot concordance group will continue to serve as a source of problems in four-manifold theory and as a testing ground for the strengths and limitations of our understanding.

1.1. Book Outline

We conclude this introduction with a chapter-by-chapter summary of the book.

- Chapter 2 sets up notation and presents background material regarding classical knot theory. We carefully explain orientation issues, since these are essential in concordance and can arise in subtle ways in many settings. We briefly review handle decompositions of 3- and 4-dimensional manifolds. Invariants arising from Seifert matrices, such as the Alexander polynomial and signatures, are described in the last section and repeatedly used throughout the book.
- Knot concordance is defined in Chapter 3. We show that concordance classes form an abelian group. Some ancillary material introduces a special class of slice disks, *ribbon disks*, and also discusses the four-genus and concordance-genus of a knot. The chapter ends with a brief introduction to other notions of concordance, including homology concordance and double null concordance.
- The *Algebraic Concordance Group*, $\mathcal{G}$ or $\mathcal{G}^{\mathbb{Z}}$, was defined by Levine [**?, ?**] and it classifies higher dimensional concordance groups. It is a quotient of the classical concordance group $\mathcal{C}$. In Chapter 4 we follow Levine in defining $\mathcal{G}$ in terms of the non-symmetric, integer Seifert matrices, and show that it injects into a group $\mathcal{G}^{\mathbb{Q}}$ that is similarly defined over the rationals. Furthermore, by means

of an isomorphism, the rational algebraic concordance group is then translated into one defined in terms of pairs: a rational symmetric matrix representing a bilinear form, along with an isomorphism of the underlying vector space. The last section highlights the important role played by irreducible, symmetric factors of the Alexander polynomial in determining an algebraic concordance class.

- To streamline the discussion of algebraic concordance, Chapter 5 begins with general facts regarding Witt groups, followed by basic examples. Our first example is the Witt group of rational symmetric bilinear forms. There are many homomorphisms from this Witt group to cyclic groups, and each leads to knot concordance invariants. Our final example is the Witt group of isometric structures. This group is central in understanding the algebraic knot concordance group. A brief review of p–adic numbers, and a summary of Levine's invariants that provide the classification of the algebraic concordance group are also included in this chapter.
- Chapter 6 completes the purely algebraic thread that begins in Chapter 4. Here we consider the order of elements in $\mathcal{G}$. Levine proved that torsion is limited to orders two and four. We describe invariants that detect elements of both orders. In the final section of this chapter we discuss the algebraic orders of knots of 12 or fewer crossings.
- In Chapter 7 we move to geometric considerations. Associated to a knot there are finite cyclic covering spaces, cyclic branched covering spaces, and the infinite cyclic cover. We discuss properties of these covers and their relationship to the algebraic invariants of a knot introduced in Section 2.10. In Section 7.10 we study the homology and cohomology of branched covers of the 4–ball over surfaces. These constrain the algebraic invariants of covers of knots in terms of the surfaces they bound in the 4–ball; in particular, they limit the possible values of invariants of slice knots.
- In 1973, Casson and Gordon proved that unlike in higher dimensions, the surjection of $\mathcal{C}$ to $\mathcal{G}$ has a nontrivial kernel. This revealed the concordance group as one of the places at which high-dimensional knot theory and classical knot theory diverge. Chapter 8 is focused on discussing Casson-Gordon invariants. Of particular interest is to realize these as bordism invariants, which also proves useful in the next chapter in showing non-triviality of certain concordance classes.

- Chapter 9 concerns the 4-torsion subgroup of $\mathcal{G}$. We discuss some of our own work, where we proved that certian invariants, which on one hand demonstrate that a class belongs to four-torsion in $\mathcal{G}$, specifically obstruct some knots in that class from being of order four in $\mathcal{C}$.
- We conclude with Chapter 10, in which we discuss some other issues related to smooth and topological locally flat concordance, including . . . . .

CHAPTER 2

Knots and surfaces

In this chapter, we review relevant notions from the study of 3 and 4 dimensional manifolds and knot theory, and establish notation. Outlines of proofs are given in some places, and references are included for details.

2.1. Manifolds, orientations and embeddings

We denote by $\mathbb{R}^n$ Euclidean n–space with the standard topology generated by the metric

$$|x| = \sqrt{\sum_{i=1}^{n} x_i^2}, \text{ for } x = (x_1, \ldots, x_n), \text{ and } d(x, y) = |x - y|.$$

An orientation on $\mathbb{R}^n$ is a choice of one orbit in the set of ordered bases under the action of the connected group $\mathrm{GL}^+(n;\mathbb{R}) = \{A \in \mathrm{GL}(n;\mathbb{R}) : \det(A) > 0\}$; equivalently, an orientation corresponds to an ordering of the vectors in a given basis up to even permutations. We will usually assume that $\mathbb{R}^n$ has the standard orientation given by its canonical ordered basis. Let $\mathbb{R}^n_0$ denote the set of non-zero vectors in $\mathbb{R}^n$. Yet another equivalent formulation of orientation of $\mathbb{R}^n$ is a choice of a generator for $H_n(\mathbb{R}^n, \mathbb{R}^n_0; \mathbb{Z}) \cong \mathbb{Z}$. This preferred generator is called the fundamental class.

For the definitions of smooth manifolds, diffeomorphisms, orientations and related basics, see [**?, ?, ?, ?**]. We will give below an informal description.

According to the Whitney embedding theorem any n–dimensional smooth manifold M with boundary ∂M is a subset of some Euclidean space of a higher dimension. (See for example [**?**, Theorem 36.2] Each point of M has a neighborhood diffeomorphic either to $\mathbb{R}^n$ or to the upper half space $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. Points of M with neighborhoods diffeomorphic to $\mathbb{R}^n$ form a smooth n–dimensional manifold which is an open subset of M. The rest of the points constitute a closed subset of M which is a smooth $(n-1)$–dimensional manifold known as the boundary of M, denoted by ∂M. The boundary ∂M of a smooth, compact manifold M has a collar neighborhood; that is, there is an open neighborhood of ∂M in M which is diffeomorphic to $\partial M \times [0,1)$. A compact, connected smooth manifold without boundary is called a closed manifold.

An n–dimensional manifold with boundary has a tangent bundle with fibers isomorphic to $\mathbb{R}^n$ at each point. A fiber is the tangent space at a point. If x is a boundary point of M, then the fiber TM_x contains an $(n-1)$–dimensional subspace $T\partial M_x$ consisting of vectors which are tangent to the boundary. The hyperplane $T\partial M_x$ divides TM_x into two open subsets, one of which consists of vectors that "point outwards" from M.

Informally speaking, an orientation of a manifold with boundary is a consistent choice of orientations for the tangent spaces at the points in the manifold. If such a choice is possible, the manifold is called orientable. The orientation of M induces an orientation for ∂M by the standard convention "outward normal comes first." For a compact M, orientability is equivalent to non-triviality of $H_n(M, \partial M; \mathbb{Z})$, which in turn is equivalent to $H_n(M, \partial M; \mathbb{Z})$ being isomorphic to the the infinite cyclic group $\mathbb{Z}$. An orientation for M specifies a choice of one of the two generators, referred to also as the fundamental class. (See Appendix A of [**?**] or [**?**, Chapter XIV].) An n–dimensional submanifold of an oriented manifold inherits orientation via restriction. By $-M$ we mean the manifold M with the opposite orientaion.

If the boundary ∂M is disconnected, sometimes we will split it as $\partial_+ M \sqcup (-\partial_- M)$, where $\partial_+ M$ and $\partial_- M$ are both $(n-1)$–dimensional submanifolds of ∂M, with induced orientation as indicated, and whose disjoint union is ∂M. For example, if X is a closed, oriented manifold, it is natural to think of the boundary of the product $X \times I$ as

$$\partial(X \times [-1,1]) \;=\; (X \times \{1\}) \sqcup -(X \times \{-1\}).$$

The standard orientation on $\mathbb{R}^n$ induces an orientation on the unit n–ball

$$B^n \;=\; \{\, (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \Sigma x_i^2 \leq 1 \,\}$$

and hence its boundary, S^{n-1}. We will sometimes use D^n instead of B^n to denote the n–dimensional unit ball. In dimension one, it is the interval $[-1,1]$, which is sometimes denoted by I. It is customary to define B^0 as a point, and S^{-1} as the empty manifold.

A diffeomorphism between two n–dimensional manifolds is said to preserve or reverse orientation according to whether the induced homomorphism on homology carries the preferred generator to the preferred generator or to the other generator, respectively.

An embedding f of a manifold N into the interior of a manifold M is a diffeomorpshism of N onto the image submanifold $f(N) \subset \mathrm{Int}(M)$. If M and N are both manifolds with boundary, then N is said to be *properly embedded* in M, if $f(N) \subset M$ and $f(N) \cap \partial M = f(\partial N)$.

An n–dimensional submanifold N of an m–dimensional manifold M has a tubular neighborhood given by an embedded normal bundle. If N is a submanifold of M, (M, N) is called a manifold pair. An oriented manifold pair (M, N) is one where each manifold is given an orientation; the notation $-(M, N)$ means $(-M, -N)$. [1]

We will work in the smooth, oriented category, except where specifically noted. Thus, manifolds are smooth and oriented, maps are smooth, and diffeomorphisms are orientation preserving. We will be mostly concerned with compact, connected manifolds with or without boundary.

EXERCISE 2.1.1. Define the 3–sphere as

$$S^3 = \{ (x, y, z, w) \in \mathbb{R}^4 \,|\, x^2 + y^2 + z^2 + w^2 = 1 \} .$$

Show that it is homemorphic to

(i) The one-point compactification of $\mathbb{R}^3$,
(ii) A union of two solid tori along their boundaries where the meridian of one is identified with the longitude of the other, and
(iii) A union of two 3 dimensional balls along their boundaries.

In Section 2.2 we discuss handle decompositions of smooth manifolds, particularly in dimensions 3 and 4. These are useful in Chapter 3 when we study slice disks (Section 3.1) and their exteriors (Section 3.2).

2.2. Handle decompositions and Morse functions

[2]

One of the aims behind the study of handlebody theory is to try to understand a manifold by decomposing it into simpler pieces. The smooth and the piecewise linear (PL) handle decompositions are equivalent in dimensions ≤ 4. The introduction below is based on Sections 4.1-2 and 6.2 of [**?**]. Outlines of proofs are available in [**?**], and more details are in [**?, ?, ?**].

[1]

[2]Swatee proposes removing this section and replacing it with recollections of ideas when we need them. This would be a big revision.

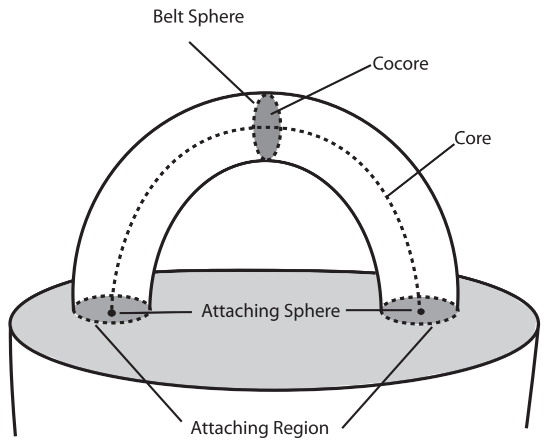

FIGURE 2.1. A 1–handle $B^1 \times B^2$ in dimension 3

2.2.1. **Definitions:** If $M \subset M'$ are n–manifolds, we say that M' is constructed from M by adding a k–handle if $M' = M \cup h_k$, where $h_k \cong B^k \times B^{n-k}$, $0 \leq k \leq n$, and $M \cap h_k \cong S^{k-1} \times B^{n-k}$. In this case, there is a corresponding embedding ϕ of $S^{k-1} \times B^{n-k}$ into ∂M called the attaching map of h_k. Alternatively, if M is an n–manifold with boundary and ϕ is an embedding of $S^{k-1} \times B^{n-k}$ into ∂M, there is a natural construction of a manifold M' containing M such that M' is constructed from M by adding a k–handle. To make this precise in the smooth category, one must "smooth corners," as described, for instance, in [**?**, Remark 1.3.3].

The core of a k–handle h_k is the image of $B^k \times 0 \cong B^k$, the boundary of the core is $\partial B^k \times 0$ called the attaching sphere of h_k, the cocore is the image of $0 \times B^{n-k} \cong B^{n-k}$, and the boundary $0 \times \partial B^{n-k}$ of the cocore is called the belt-sphere of h_k. The attaching region is the image of $\partial B^k \times B^{n-k}$ in ∂M. This terminology is depicted in Figure 2.1.

The number k is called the index of the handle and, up to homotopy, attaching a k–handle is the same as attaching a (thickened-up) k–cell. In particular, attaching a 0–handle is taking disjoint union of M with B^n, whereas attaching an n–handle can only be done with manifolds which have a boundary component diffeomorphic to S^{n-1}, since it is the same as forming a boundary connected sum with B^n.

Let M be a compact manifold with boundary $\partial M = \partial_+ M \sqcup -\partial_- M$. A handle decomposition of M relative $\partial_- M$ is an identification of M with a manifold obtained by attaching handles to $I \times \partial_- M$, where I is an interval $[0, a]$, for some $0 < a < 1$, and

$\{0\} \times \partial_- M$ is naturally identified with $\partial_- M$. A manifold M with a given handle decomposition is called a handlebody. (Some authors will use the term "relative handlebody" built on $\partial_- M$, and call M a handlebody only when $\partial_- M = \phi$.)

A Morse function on a compact n–dimensional manifold M with boundary $\partial_- M \sqcup -\partial_- M$ is a smooth function $h\colon M \to [0,1]$ with $h^{-1}(0) = \partial_- M,\ h^{-1}(1) = \partial_+ M$, where h has no critical points on ∂M and only non-degenerate critical points at various $t \in (0,1)$. Recall that p being a critical point of h means that in a coordinate system about p, we have $\frac{\partial h}{\partial x_i}(p) = 0,$ for all i. Furthermore, a critical point p being non-degenerate means that the determinant of the Hessian matrix $\left(\frac{\partial^2 h}{\partial x_i \partial x_j}(p)\right)_{i,j}$ is nonzero. The Morse lemma states that for a non degenerate, critical point p, in some coordinate system about p, we have

$$h(x_1, \ldots, x_n) = c - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2,$$

where c is a constant, and $0 \leq k \leq n$. The number k is called the index of the critical point p.

Any handle-body decomposition of M relative $\partial_- M$ corresponds to a Morse function h on M. A critical point of h of index k corresponds to a k–handle $B^k \times B^{n-k}$. See Figure 2.7; [**?**, **?**] provide more details.

2.2.2. **Handle modifications:**

It is a general fact that a handle decomposition can be modified by isotoping attaching maps, so that handles are attached in order of increasing index [**?**, Proposition 4.2.7].

A $(k-1)$–handle h_{k-1} and a k–handle h_k can be "cancelled," provided that the attaching sphere of h_k intersects the belt-sphere of h_{k-1} transversely once [**?**, Proposition 4.2.9], as seen in Figure 2.2.

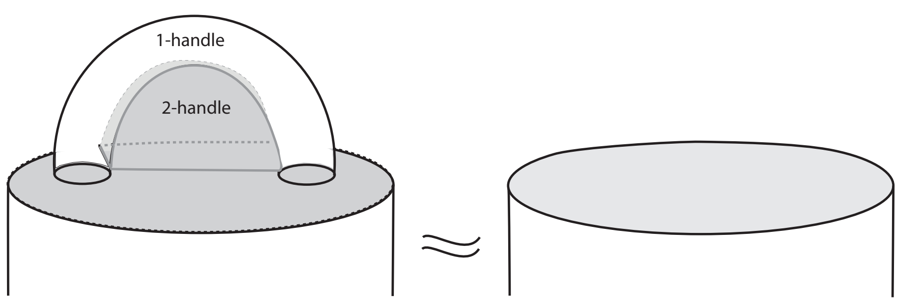

FIGURE 2.2. A canceling handle pair in dimension 3

Using handle cancellation in reverse we can see that removing a j–handle can be seen as the addition of a $(j \pm 1)$–handle.

2.2.3. **Turning a handlebody upside-down:**

A handle decomposition of M, relative $\partial_- M$, can be turned "upside down" to obtain a dual decomposition relative $-\partial_+ M$. Instead of $I \times \partial_-(M)$, we now begin with $I \times (-\partial_+(M))$. A k–handle $B^k \times B^{n-k}$ in the handle decomposition relative $\partial_-(M)$ now corresponds to an $(n-k)$–handle $B^{n-k} \times B^k$ in the dual decomposition relative $-\partial_+(M)$ reversing the roles of core and cocore. This process preserves the property of handles being attached in the order of increasing index. In terms of Morse theory, this is equivalent to replacing a Morse function h by $1 - h$.

2.2.4. **Dimensions three and four:**

Our interest lies mainly in dimensions three and four. As adding either a 0–handle or a top dimension handle is fairly straight forward, we will take a quick look at the remaining cases.

In dimension 3, for attaching a 1–handle $B^1 \times B^2$, we need to specify an embedding of $S^0 \times B^2$; i.e., we identify two disjoint embeddings of a 2–disk in the boundary of the 3–manifold. A 2–handle $B^2 \times B^1$ is attached along an annulus $S^1 \times B^1$ in the boundary.

In dimension 4, a 1–handle $B^1 \times B^3$ is attached along two 3–balls embedded in the boundary. This is usually indicated using one of the diagrams in Figure 2.3 below. In the dotted circle notation, we are using the fact that addition of a 1–handle is equivalent to the removal of a (canceling) 0–handle. If the obvious disk bounded by this circle is pushed inside the 4–ball, by a map $(B^2, S^1) \to (B^4, S^3)$, and its neighborhood removed, we get an $S^1 \times B^3$. As the attaching sphere $S^0 \times 0$ of the 1–handle and the belt-sphere $0 \times S^3$ of the 0–handle should intersect in one point; any arcs in the resulting manifold that go over the 1–handle should be drawn so as to pass through the dotted circle.

Next, for attaching a 2–handle $B^2 \times B^2$ in dimension 4, the attaching region is a solid torus $S^1 \times B^2$. Thus, to specify how to attached a 4-dimensional 2-handle, we start with a circle K (typically knotted) and then must give an identification of a neighborhood of that circle with $S^1 \times B^2$. In order to do so, we indicate to which pushoff of K this identification will send $S^1 \times \{1\}$. If K is nullhomologous, then we call the unique pushoff which will be nullhomologous in the complement of K the 0-framed pushoff or 0-framed logitude. For example see Figure 2.4 for the 0-framed pushoff of the unknot. For any $f \in \mathbb{Z}$ we call

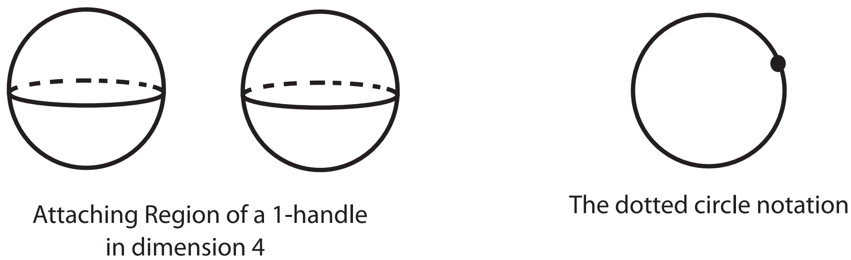

FIGURE 2.3. Attaching a 1–handle in dimension 4

the f-framed pushoff the one which is homologous to f copies of the meridian of K. In order to specify a 2-handle, we will label a knot in B^4 with an integer. We will be mostly concerned with the 0 framing in this section and in Chapter 3. See [**?**] for more on this notation in the context of 4 dimensional manifolds[3].

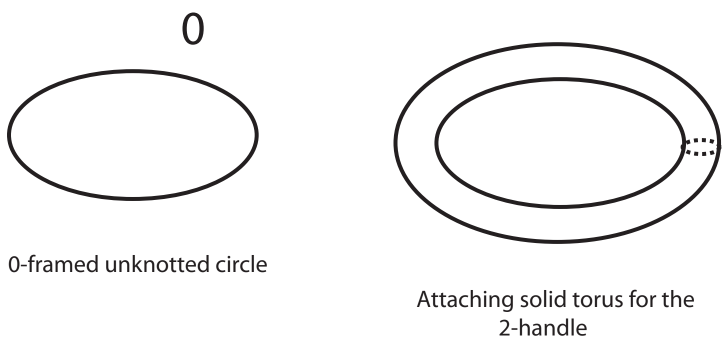

FIGURE 2.4. Attaching a 2–handle in dimension 4

A canceling pair of a 1–handle and a 2–handle looks like in Figure 2.5 below.

Handles can be modified through a technique known as handle-sliding which does not change the diffeomorphism type of the manifold. This procedure is illustrated in Figure 2.6 below where we slide a 2–handle over a 1–handle, and then erase a canceling handle pair. The unlabeled double arrow indicates an orientation preserving diffeomorphism between the two manifolds.

[3]A handlebody description of a 4–manifold with boundary gives a surgery description of the boundary 3–manifold. Framings in the context of knot surgery and 3–manifolds are discussed in 2.6.9.

FIGURE 2.5. A canceling handle pair of a 1– and a 2–handle in dimension 4.

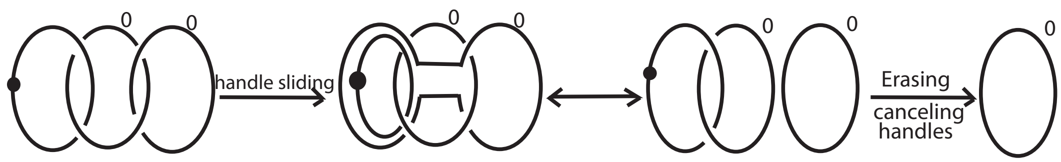

FIGURE 2.6. Sliding a 1–handle over a 2–handle

THEOREM 2.2.5. [**?**, Theorem 4.2.12] Given any two relative handle decompositions of a compact pair $(M, \partial_- M)$, ordered by increasing index, one can get from one to the other by a sequence of handle slides, creating or annihilating canceling handle pairs and isotopies.

Note that adding a 1–handle to a 4–ball results in $S^1 \times B^3$ with boundary $S^1 \times S^2$. On the other hand, adding a 2–handle (along a 0-framed unknot) results in $S^2 \times B^2$ also with boundary $S^1 \times S^2$. Handles that are attached later along the boundary $S^1 \times S^2$ do not require information about the manifold this $S^1 \times S^2$ bounds. In this sense, when the interest is only in studying the boundary, sometimes dotted circles (1–handles) are replaced by unknots with framing 0 (2–handles).

2.2.6. **Handle decomposition of an embedding of a 2–disk in dimension 4**

In Chapter 3, Sections 3.1 and 3.2, we need to consider handle decompositions of disks properly embedded in the 4–ball and exteriors of such disks in the 4–ball. We now consider this situation. Our discussion is modeled after [**?**, §6.2].

Figure 2.7 below shows a two dimensional manifold D diffeomorphic to a 2–dimensional disk along with a Morse function to $[0, 1]$ given by the "height" function h. Critical points of h correspond to function values a, b, c, d, e and these have been arranged in the order of their index.

The handle decomposition of D given by the Morse function is easy to see. Our starting place, $\partial_- M$, is the empty set. The index 0 critical points of h corresponding to

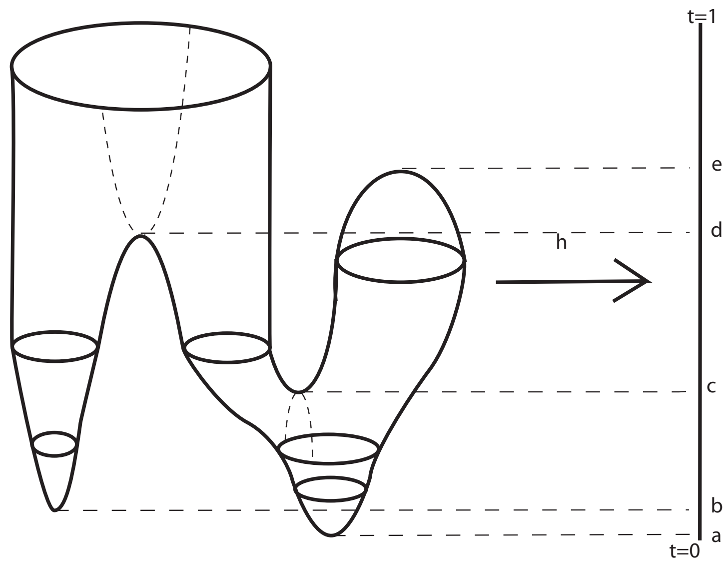

FIGURE 2.7. A Morse function on a 2–dimensional disk

values a and b are local minima of the function and both contribute 0–handles which in this case are 2–dimensional disks. The index 1 critical points at c and d are saddle points in the manifold, and the index 2 critical point at height e is a local maximum. Between any 2 critical values the manifold has a product structure. For instance, $h^{-1}(a, b)$ is diffeomorphic to an annulus $S^1 \times (a, b)$.[4]

In Figure 2.8 we see what $h^{-1}(c, d]$ looks like. First, we have the product $h^{-1}(c, d) = h^{-1}\{p\} \times (c, d)$, where p is a point in the interval (c, d). This is diffeomorphic to a disjoint union of three annuli. An index 1 handle is added at the critical value d of h which has Morse index 1. The addition of a 1–handle results in a saddle point for the embedding of D.

FIGURE 2.8. A 2–dimensional 1–handle

4

Between $t = d$ and $t = e$, we see a disjoint union of two annuli. Finally the critical point corresponding to the value e is a local maximum of the function h and it contributes an index two handle which is a 2–disk attached along a circle in the boundary of the manifold constructed up to that point, thus capping off one of the bounding circles. At $t = 1$ we see the boundary of the 2–manifold D to be a circle.

2.2.7. **Disk exterior in a 4-ball**

[5]Let D be the 2-disk in Figure 2.7. Assume that D is properly embedded in a 4-ball B, i.e., $\partial D = D \cap \partial B$. We will further assume that the Morse function h is the radius function on B restricted to D, that the critical points of h are arranged in increasing order of their Morse index with critical values a, b, c, d, e as shown in Figure 2.7, and that a k-handle of D is flattened to lie in a single level as in Figure 2.8. Let $N(D)$ denote a tubular neighborhood of D. The disk exterior $X = X(D)$ is defined to be $B - \mathrm{Int}(N(D))$. The exterior X inherits a handle decomposition from D which we describe below.

For notational convenience, for $t \in (0, 1]$, let B_t denote the 4-ball of radius t centered at the origin, let $D_t = D \cap B_t$ and $X_t = X \cap B_t$. We mentioned earlier that on an interval $[x, y] \subset (0, 1]$ which contains no critical value of h, the embedding of D is a product up to isotopy.

We will now build the exterior X starting with small values of t and moving upwards. At the beginning, we have a 4-ball $B_{a'}$ of radius $a' < a$. As we pass $t = a$, we see the addition of a 4-dimensional 1-handle $B^1 \times B^3$ attached along $S^0 \times B^3$. In general, as t increases, corresponding to each k-handle of D, the topological type of X_t changes by attachment of a handle. Proposition 6.2.1 of [**?**] tells us that if $[u, v] \subset (0, t]$ is an interval containing a unique critical value $w \in (u, v)$ of the Morse function h which corresponds to a unique k-handle of the 2-disk D with core B^k, then $X_v \cong X_u \cup$ $(k + 1)$-handle with core $B^k \times B^1$. A key step in the proof is to use duality: removing a handle can be seen as adding a canceling handle (see Figure 2.2 and the discussion in 2.2.2).

In Figure 2.9, the two dotted circles denote 1-handles $B^1 \times B^3$ attached along $S^0 \times B^3$. These correspond to the removal of a 4-ball or a 0-handle (neighborhood of a 2-disk) at $t = a$ and $t = b$, respectively. Removing a neighborhood of the attaching band for the index 1 critical point ($t = c$) from B corresponds to attaching a 2-handle $B^2 \times B^2$ to X_t, along an embedded solid torus $S^1 \times B^2$ whose core is the curve labeled with framing 0.

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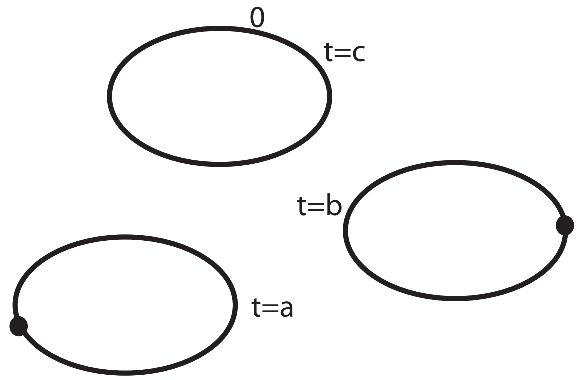

FIGURE 2.9. Building the exterior of a disk in the 4-ball; $t \in [a, c]$.

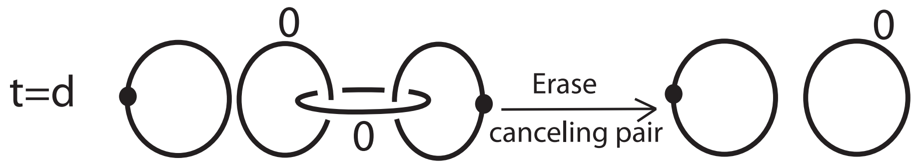

FIGURE 2.10. Building the exterior of a disk in the 4-ball: handle slide

The change in X_t at level $t = d$ is a bit different. As in the case of $t = c$ above, we are adding a 2-handle $B^2 \times B^2$ attached along $S^1 \times B^2$, but it now goes over a previously attached 1-handle. The 2-handle at $t = d$ looks like the picture on the left in Figure 2.10. Sliding the left circle with 0-framing over the dotted circle and then erasing the canceling pair of a 1- and a 2- handle as in Figures 2.5 and 2.6, we obtain the picture on the right.

As discussed earlier, the two circles on the right in Figure 2.10 each contribute a boundary component diffeomorphic to $S^2 \times S^1$. At $t = e$, X_t changes by addition of a 3-handle $B^3 \times B^1$ which "caps off" one of these boundary components. The resulting manifold is the disk exterior in B, and its boundary is diffeomorphic to $S^1 \times S^2$.

[6]

EXERCISE 2.2.8. Using handle-slides and canceling 1- and 2-handles, show that the 4-manifolds shown in the following two pictures have the same boundary three-manifold.

EXERCISE 2.2.9. Find the first homology group of the boundary 3-manifold of the 4-manifolds described in Figure 2.11.

[6]. Swatee suggests maybe making some major removals in this section.

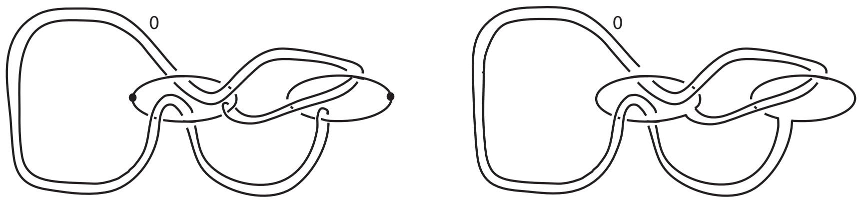

FIGURE 2.11.

2.3. Knots in S^3

Our main objects of study are embeddings of 1-dimensional circles in the 3-sphere S^3. Standard references for information in the rest of this chapter include [**?**, **?**, **?**, **?**]

DEFINITION 2.3.1. A (classical) knot is an oriented diffeomorphism class of a pair of oriented manifolds, (M, K), with M and K oriented diffeomorphic to S^3 and S^1, respectively. As shorthand, such a pair is denoted simply by K and we write S^3 instead of M unless there is a reason to be explicit.

The orientation of a knot is sometimes referred to as the string orientation to distinguish from the orientation of the ambient space.

Knots K_1 and K_2 are isotopic if there is a map $f : S^1 \times [0, 1] \to S^3$ called an isotopy such that $f(S^1 \times \{0\}) = K_1$, $f(S^1 \times \{1\}) = K_2$ (orientation will be opposite), and $f|_{(S^1 \times x)}$ is an embedding for all $x \in [0, 1]$. By the isotopy extension theorem, this is equivalent to the existence of an ambient isotopy $F : S^3 \times [0, 1] \to S^3$ from the identity of S^3 to a map g such that $g(K_1) = K_2$.

It is easy to see that this is an equivalence relation. That is, it is reflexive, symmetric and transitive. Another formulation of knot isotopy is that there is an orientation preserving diffeomorphism f of S^3 to itself, such that $f(K_1) = K_2$. These two formulations are equivalent since any orientation preserving diffeomorphism of S^3 is isotopic to the identity. See Proposition 1.10 of [**?**].

A link in M is an isotopy class of embeddings of a disjoint union of circles into S^3, $L : S^1 \times \{1, \dots k\} \to M$. The j'th component of L is the knot given by $L_j = L|_{S^1 \times \{j\}}$. Except when needed, we will suppress the ordering and orientation of the components of links.

Any knot (or link) in S^3 is isotopic to a knot that misses the north pole, and thus stereographic projection determines a knot in $\mathbb{R}^3$. If knots K_1 and K_2 are isotopic in S^3, the corresponding knots in $\mathbb{R}^3$ are isotopic. In fact, there is a bijection between the set of isotopy classes of knots in S^3 and in $\mathbb{R}^3$. This permits us to illustrate a knot in S^3 using a *knot diagram*, the image of a regular projection of a knot in $\mathbb{R}^3$ to the (x, y)–plane in $\mathbb{R}^3$, with gaps in the projection indicating which of the two points mapping to a double point has a smaller z–coordinate. [7] Every knot is isotopic to a knot having a regular projection, and two knots with the same diagram are isotopic. In Figure 2.12 we see two different diagrams both representing the trefoil knot.

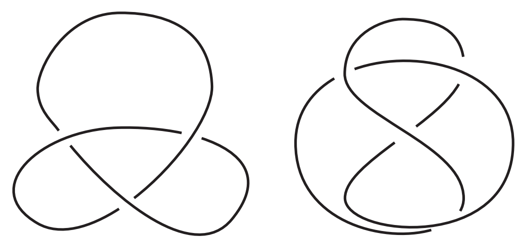

FIGURE 2.12. Two diagrams representing the trefoil knot

It is customary to draw "unoriented" knot diagrams, as in Figure 2.12. However, for our main topic of interest in this book, namely, concordance of knots, orientations are essential. We will specify the string orientation of a knot when needed. We now establish notation for knots obtained by changing orientations.

DEFINITION 2.3.2. Given a knot (M, K), one can form three other knots: the mirror image or the obverse $m(M, K) = (-M, K)$, the reverse $r(M, K) = (M, -K)$, and the inverse $-(M, K) = r \circ m(M, K) = (-M, -K)$. Notice that the mirror image and the inverse are knots in $-M$, not in M. It is standard to abuse notation and suppress then 3-manifold, M from the notation writing $r(K)$, $m(K)$ and $-K$.

There is not consistent usage of these terms in the literature; in particular, the reverse of a knot is sometimes called its inverse. In Chapter 3, we will see that $-K$ represents the inverse of the equivalence class of K in the abelian group of knot concordance classes (see Theorem 3.3.3 (3)), and this justifies the usage we have adopted. Note also that our definition of $-K$ in this case is consistent with the definition of the manifold pair with opposite orientations, in Section 2.1, as by a knot K, we mean the manifold pair (S^3, K).

When working with knots in S^3, it is natural to consider the mirror image and the inverse as knots in S^3; this can be accomplished as follows. Let $F : S^3 \to S^3$ be an

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orientation preserving diffeomorphism from $-S^3$ to S^3. (We will typically choose F so that restricted to $\mathbb{R}^3$, F becomes reflection in a coordinate plane.) Then $F(mK)$ and $F(-K)$ are knots in S^3. Every orientation preserving diffeomorphism of S^3 is isotopic to the identity, so that the choice of F does not matter. Henceforth, when working with knots in S^3, we will view each of the four knots K, mK, rK, and $-K$ as a knot in S^3.

In Figure 2.13 we illustrate the knot 8_{17} and its mirror image formed by reflecting in the (x, y)–plane. In Figure 2.14, this is illustrated schematically. The dashed diagram represents a knot with the same projection as the undashed version, but with all crossings reversed. The mirror image under the reflection through the (y, z)-plane would consist of the horizontal mirror image of J, again dashed.

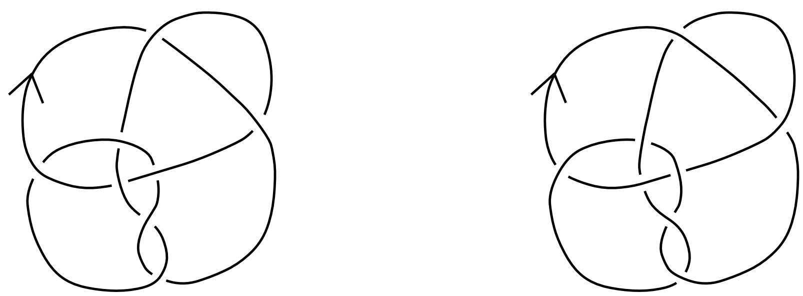

FIGURE 2.13. Knot 8_{17} and $m8_{17}$.

FIGURE 2.14. Schematic of a knot and its mirror image.

EXERCISE 2.3.3. There are 16 possible schematic diagrams of the type illustrated in Figure 2.14. The J can be placed in four possible positions (unchanged, rotated 180° about the origin, reflected over the x-axis and over the y-axis), bold or dashed, and the arrow (string orientation) can be pointed up or down. Given that the first represents a knot K, what do each of the other 15 represent?

DEFINITION 2.3.4. A knot K in S^3 is called positive amphichieral if K and mK are isotopic; K is called negative amphichieral if K and $-K$ are isotopic; it is called reversible if K and rK are isotopic. A knot is called amphicheiral if it is either positive or negative amphicheiral.

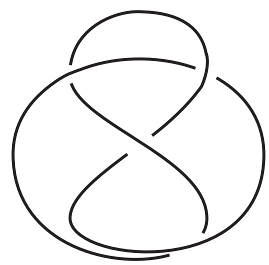

FIGURE 2.15. Figure 8 Knot or 4_1

EXERCISE 2.3.5. Show that the Figure 8 knot seen in Figure 2.15 is both positive and negative amphicheiral by demonstrating a sequence of moves in $\mathbb{R}^3$ that transform this figure into that of mK and rmK, respectively.

2.4. Connected sums

Given oriented, connected n–manifolds M_1 and M_2, the connected sum $M_1 \# M_2$ is defined as follows. Let F_1 and F_2 be embeddings of the open n-ball $\overset{\circ}{B}{}^n$ into M_1 and M_2, respectively. Then $M_1 \# M_2$ is formed as the quotient of the disjoint union of $M_1 - F_1(0)$ and $M_2 - F_2(0)$ as follows. Identify $\overset{\circ}{B}{}^n - 0$ with $(0,1) \times S^n$, let $g\colon S^n \to S^n$ be an orientation reversing map given by reflection in the $(x_2, x_3, \dots, x_n)$ hyperplane, and now identify $F_1(r \times x)$ with $F_2((1-r) \times g(x))$, for $0 < r < 1$.

For knots K_i in M_i, the connected sum $K_1 \# K_2$ is formed similarly, where now in place of the $F_i(\overset{\circ}{B}{}^n)$ we have standard (open) ball pairs, $(\overset{\circ}{B}{}^3, \overset{\circ}{B}{}^1)$. Moreover, $S^3 \# S^3$ is diffeomorphic to S^3 and any two such diffeomorphisms are isotopic. (The analogous statement is false in higher dimensions, but there is a canonical choice of an isotopy class of diffeomorphisms from $S^n \# S^n$ to S^n.) Thus, we view the connected sum of knots in S^3 as again being in S^3.

Equivalently, sometimes for convenience a connected sum is viewed as the adjunction space

$$\overline{(M_1 - B_1)} \cup_{\partial B_1 \cong -\partial B_2} \overline{(M_2 - B_2)},$$

where the B_i are embeddings of a closed n-ball (or ball pairs) in the manifolds M_i (or manifold pairs) and the bar denotes closure.

REMARK 2.4.1. In order for the connected sum of two knots to be a well-defined knot, attention to the string orientations of component knots is important. Connected sum is a well-defined binary operation on (oriented) isotopy classes of knots. It is commutative, associative, and the class of the unknot serves as the identity element. (See Lemma 7.3 in [?].) However, we cannot form a connected sum of a non-trivial knot with another knot to get the unknot, so we have no inverses. For instance, knot genus (defined in 2.7.6) adds under connected sum (see 2.7.7), and the genus of a non-trivial knot is a positive integer, making cancellation impossible.

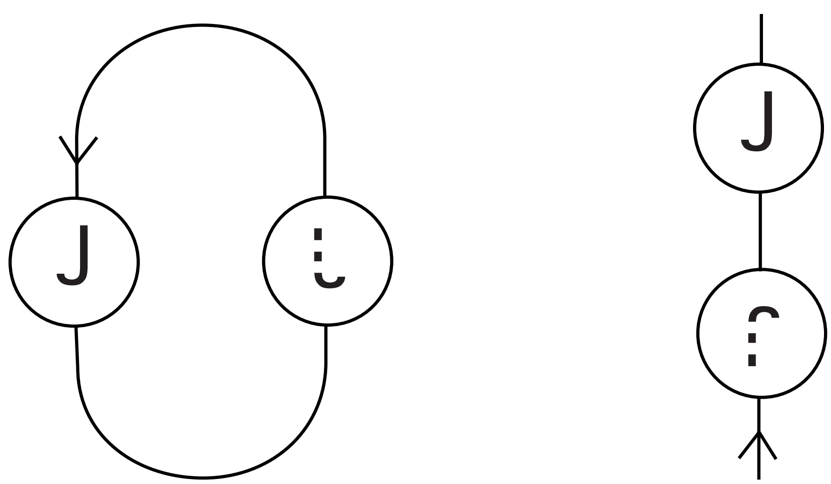

FIGURE 2.16.

EXERCISE 2.4.2. Figure 2.16 presents schematic illustrations of two knots, each of the form $K_1 \# K_2$. Express K_1 and K_2 in terms of K where K is represented by the schematic built from J alone.

DEFINITION 2.4.3. A knot is called prime, if it cannot be expressed as the connected sum of two non-trivial knots.

Equivalently: Consider an embedded 2-sphere S in S^3 which intersects a knot K in two points. Then $S^3 - S$ has two path components. The knot K is the union of two arcs, say, A_1 and A_2 with end points on S and interiors inside different components. Draw an arc A_S on the sphere joining the two points which constitute $S \cap K$. The knot K is prime, if and only if given any such sphere S and arc A_S, either $A_1 \cup A_S$ or $A_2 \cup A_S$ is necessarily a trivial knot.

Torus knots $T(p, q)$, depicted in Figure 2.17 , where p, q are relatively prime integers, are prime. For a proof of this and for more on connected sums, see [**?**, Chapter 7, §A] and [**?**, Chapter 2, §G]. Standard knot tables, as in [**?, ?, ?**], list one representative each of isotopy classes of prime knots without reference to orientation either of the knot or of S^3.

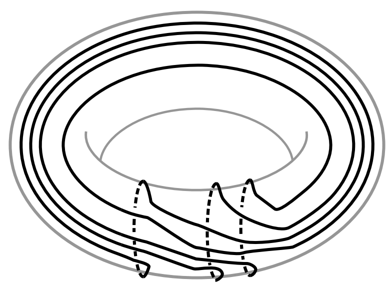

FIGURE 2.17. The (p, q) torus knot $T_{p,q}$ is the know sitting on the unknotted torus in S^3 winding p times in the longitudinal direction and q times in the meridonal direction. Depicted is the $(4, 3)$-torus knot.

2.5. Tubular neighborhoods and knot complements

[8]Let $V = S^1 \times B^2$, with meridian $m = \{(1, 0)\} \times \partial B^2$ and longitude $l = S^1 \times \{(1, 0)\}$. Being a submanifold, a knot K in S^3 has a tubular neighborhood $N(K)$ which is the image of an embedding $f : V \to S^3$ satisfying $f(S^1 \times (0, 0)) = K$. This neighborhood is unique up to an isotopy that fixes K. The image $f(m)$ is called the meridian of K. It is unique up to isotopy on $\partial N(K)$. Note, however, that the isotopy class of $f(l)$ depends on the choice of f. There is a self-diffeomorphism h of V given by $h((x, y)) = (x, xy)$, where the product xy is defined via the identification of $\mathbb{R}^2$ with $\mathbb{C}$. Composing f with the n–fold composition h^n gives a new embedding f_n. Every embedding g, with $g(S^1 \times 0) = K$, is

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isotopic to exactly one of the f_n (where n might be negative), and n is determined by the isotopy class of $g(l)$ on $\partial N(K)$.

The complement of a knot K, is simply $S^3 - K$. The exterior of K is defined to be $X(K) = S^3 - \overset{\circ}{N}(K)$, a closed manifold with boundary $\partial X(K) = \partial N(K) \cong \mathbb{T}^2 \cong S^1 \times S^1$. It is clear that knots with diffeomorphic exteriors have diffeomorphic complements. The converse is true, but it is nontrivial. In fact, a much stronger result is known: If the knot complements are diffeomorphic, then the corresponding knots are isotopic. See [**?**].

Remark 2.5.1. The knot group $\pi_1(S^3 - K) = \pi_1(X(K))$ is a powerful knot invariant in that prime knots with isomorphic knot groups have diffeomorphic complements [**?**]. However, it is not easily computable, and its abelianization $H_1(S^3 - K)$ is the infinite cyclic group $\mathbb{Z}$ for all knots, as seen in Theorem 2.5.2 below.

As discussed above, each identification of $N(K)$ with V gives a different choice of longitude. We explain how to specify a well defined preferred longitude for a knot K in a closed, oriented 3-manifold M without boundary, provided K represents $0 \in H_1(M; \mathbb{Q})$. In the remainder of this and the following section, we briefly discuss this general case. However, the reader may wish to ignore the generality and consider M to be S^3.

The homology exact sequence of the pair $(M, X(K))$ with coefficients in $\mathbb{Z}$ yields:

$$H_2(M) \to H_2(M, X(K)) \to H_1(X(K)) \to H_1(M) \to H_1(M, X(K)) \to 0.$$

Via excision and duality, this can be rewritten as:

$$H_2(M) \to \mathbb{Z} \to H_1(X(K)) \to H_1(M) \to 0.$$

Here are three consequences.

Theorem 2.5.2. Let M be a closed 3-manifold, and let K be a knot in M.

(1) If M is an integral homology 3-sphere; i.e. if $H_*(M) = H_*(S^3)$, then $H_1(X(K)) \cong \mathbb{Z}$. Using orientation there is a canonical isomorphism.
(2) If M is a rational homology sphere; i.e. if $H_*(M; \mathbb{Q}) = H_*(S^3; \mathbb{Q})$, then there is a short exact sequence:

$$0 \to \mathbb{Z} \to H_1(X(K); \mathbb{Z}) \to H_1(M; \mathbb{Z}) \to 0$$

which may or may not be split.

(3) If M is a rational homology sphere and K is null homologous in M; i.e. $K = 0$ in $H_1(M;\mathbb{Z})$, then there is a split exact sequence:

$$0 \to \mathbb{Z} \to H_1(X(K);\mathbb{Z}) \to H_1(M;\mathbb{Z}) \to 0.$$

REMARK 2.5.3. A useful perspective on the homology of $X(K)$ is obtained by observing that M is built from $X(K)$ by adding a 2–cell and a 3–cell, since the solid torus is built from its boundary in the same way, by first adding a 2-cell attached along a meridian of K and then filling in a 3-cell.

Proof of Theorem 2.5.2: Statements (1) and (2) are automatic, noting in the case of (2) that $H_2(M;\mathbb{Z}) \cong H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(H_1(M);\mathbb{Z}) = 0$, since $H_1(M;\mathbb{Q}) = 0$ implying that $H_1(M;\mathbb{Z})$ is torsion. See Exercise 2.5.5 below for examples which illustrate that the short exact sequence in (2) may or may not split. For the third statement, since K is null-homologous, it bounds a chain F in M. Intersections of curves in $X(K)$ with F define a homomorphism $H_1(X(K)) \to \mathbb{Z}$ taking value 1 on m. This gives the desired splitting $H_1(X(K)) \to \mathbb{Z}$. □

EXAMPLE 2.5.4. Consider the knot K' in the solid torus V illustrated in Figure 2.18.

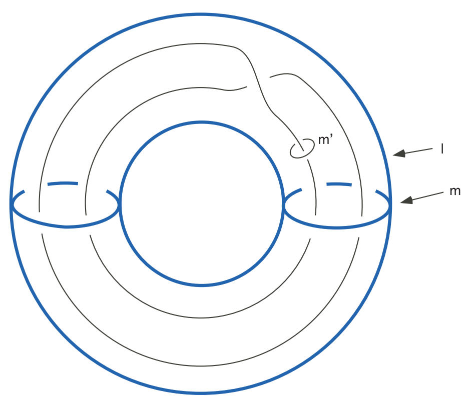

FIGURE 2.18. A knot in a solid torus

We denote the meridian and longitude of V by m and l. The meridian to K' we denote by m'. Then $H_1(V - K') \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by m' and l. Note also that $m = 2m' \in H_1(V - K')$. We can select a curve l' on $\partial N(K')$ which meets m' in one point and represents $(0, 2) = 0m' + 2l \in H_1(V - K')$.

Form a closed manifold M as the boundary union of V with another copy of $S^1 \times B^2$. Suppose that for a point $p \in S^1$ the curve $p \times \partial B^2$ is identified with a curve representing $am + bl \in H_1(\partial V)$. Then $H_1(X(K')) = H_1(M - K') \cong \mathbb{Z} \oplus \mathbb{Z}/\langle(2a, b)\rangle$.

Attach a solid torus $U = S^1 \times B^2$ to the exterior of K' in M. [9] Being a diffeomorphism from $S^1 \times S^1$ to itself, the attatching map induces an ismorphism of $\mathbb{Z} \times \mathbb{Z} \cong \pi_1(S^1 \times S^1)$. Thus, it sends the meridian of U to $cm' + dl'$ and the longitude maps to $em' + fl'$ for some c, d, e, f with $cd - de = 1$. Denote the resulting closed manifold by N; the core of the attached solid torus U represents a knot $J \subset N$. The meridian to J we denote by m'', and the image of the longitude of the solid torus we denote l''.

The homology of N is generated by m' and l, and with respect to these generators we have $H_1(N) \cong \mathbb{Z} \oplus \mathbb{Z}/\langle(2a, b), (c, 2d)\rangle$. With respect to the same basis, J represents $(e, 2f)$.

Using the same basis, $H_1(X(J)) \cong \mathbb{Z} \oplus \mathbb{Z}/\langle(2a, b)\rangle$; m'' represents $(c, 2d)$ and l'' represents $(e, 2f) \in H_1(X(J))$.

Exercise 2.5.5. Using the example above, select values of a, b, c, d, e, and f to do the following:

(i) Build examples for which the short exact sequence in 2.5.2 (2)

$$0 \to \mathbb{Z} \to H_1(X(J)) \to H_1(N) \to 0$$

is not split, and other examples for which it is split.

(ii) In which cases is N a rational homology sphere?

(iii) Build examples for which $\mathrm{Ker}\{H_1(\partial X(J)) \to H_1(X(J)\}$ contains a primitive element; i.e., an element which is not the integer multiple of another, and other examples for which it does not contain a primitive element.

2.6. Peripheral homology structure and knot surgery

The previous discussion motivates a consideration of the inclusion $\partial X(K) \to X(K)$. We begin with a general result.

Theorem 2.6.1. If W^{2n+1} is a compact orientable manifold, then for any coefficient field $\mathbb{F}$, we have $2 \cdot \dim(\mathrm{Ker}\{H_n(\partial W; \mathbb{F}) \to H_n(W; \mathbb{F})\}) = \dim H_n(\partial W; \mathbb{F})$. The intersection pairing (dual to the cup product pairing) $H_n(\partial W; \mathbb{F}) \times H_n(\partial W; \mathbb{F}) \to \mathbb{F}$ vanishes on this kernel.

Proof Consider the exact sequence:

$$H_{n+1}(W; \mathbb{F}) \to H_{n+1}(W, \partial W; \mathbb{F}) \to H_n(\partial W; \mathbb{F}) \to H_n(W; \mathbb{F}) \to H_n(W, \partial W; \mathbb{F}).$$

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Poincaré-Lefschetz duality [**?**, Chapter XIV, Theorem 7.7] tells us

$$H^k(W,\partial W;\mathbb{F}) \cong H_{2n+1-k}(W;\mathbb{F}), \text{ and } H^k(W;\mathbb{F}) \cong H_{2n+1-k}(W,\partial W;\mathbb{F}).$$

This isomorphism is defined by taking the cap product with the fundamental class in $H_{2n+1}(W,\partial W;\mathbb{Z})$. Also by Poincaré duality, we have $H_n(\partial W;\mathbb{F}) \cong H^n(\partial W;\mathbb{F})$, and since $\mathbb{F}$ is a field, we have the identification of $H^k(X;\mathbb{F})$ with the dual space $\mathrm{Hom}(H_k(X;\mathbb{F});\mathbb{F}) = (H_k(X;\mathbb{F}))^*$. With this in mind, the above exact sequence can be written as the following exact sequence of vector spaces over the field $\mathbb{F}$.

$$V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3 \cong V_3^* \xrightarrow{g^*} V_2^* \xrightarrow{f^*} V_1^*.$$

From exactness at $V_3 \cong V_3^*$ and the fundamental theorem of homomorphism of vector spaces, we have $V_3 \cong \mathrm{Im}(g^*) \oplus \mathrm{Im}(g)$. Next, $\mathrm{Im}(g^*) \cong \mathrm{Ker}(f^*) \cong \mathrm{CoKer}(f)$. By exactness at V_2, this cokernel is isomorphic to $\mathrm{Im}(g)$.

To check that the intersection form vanishes on the kernel, we work dually and check that the cup product pairing vanishes on the image of $H^n(W)$ in $H^n(\partial W)$. This follows immediately from the fact that the map $H^{2n}(\partial W) \to H^{2n}(W)$ is the 0 function. □

Theorem 2.6.2. Let K be a knot in the closed manifold M. There is a simple closed curve $l \subset \partial X(K)$ that generates $\mathrm{Ker}\{H_1(\partial X(K);\mathbb{F}) \to H_1(X(K);\mathbb{F})\}$. Let m denote a meridian to K. If $K = 0 \in H_1(M;\mathbb{Q})$ then $l \cap m \neq 0$ in $H_0(\partial X(K);\mathbb{Z})$. If $K = 0 \in H_1(M;\mathbb{Z})$ then $l \cap m$ is a generator of $H_0(\partial X(K);\mathbb{Z}) \cong \mathbb{Z}$. [10]

By picking an orientation of $\partial X(K)$ we get a preferred identification of $H_0(\partial X(K);\mathbb{Z})$ with $\mathbb{Z}$.[11] In the case that K is 0 in $H_1(M;\mathbb{Q})$ we select l so that $l \cap m > 0 \in \mathbb{Z}$ and call this l a longitude of K. When K is 0 in $H_1(M;\mathbb{Z})$, the longitude l is the null-homologous longitude described in Remark 2.6.6 below.

Exercise 2.6.3. In Examples 2.5.4 and 2.5.5, identify the longitude of J, when defined.

Exercise 2.6.4. Use appropriate choices of a, b, c, d, e, and f in 2.5.5 to construct explicit examples illustrating the range of possibilities for the structure of the homology of the manifold N and the knot exterior $X(J)$ in N.

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2.6.5. **Linking numbers in S^3**

As we are discussing framings, this is a good place to introduce linking numbers.

The linking number $\mathrm{lk}(\alpha, \beta)$ of disjoint, oriented, simple closed curves α and β in S^3, is defined to be the class represented by β in $H_1(X(\alpha))$, thought of as an element of $\mathbb{Z}$. Recall that $X(\alpha)$ is the exterior of the curve α as defined immediately prior to Remark 2.5.1. The linking number can be computed algorithmically by counting the number of times β passes over α, counting the crossing positive or negative if α passes under from right to left or left to right, respectively.

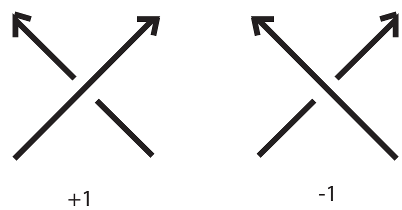

FIGURE 2.19. Sign convention in linking numbers

If the curves are not embedded, the linking number is defined by taking small perturbations so that each becomes embedded. It is the case that $\mathrm{lk}(\alpha, \beta) = \mathrm{lk}(\beta, \alpha)$. This symmetry is most easily seen by considering the algorithm for computing linking numbers and noting that one could also count at the under-crossings rather that at the overcrossings. Therefore, linking number is well-defined for two-component links. The linking number between two homology classes is defined to be the linking number between their representative curves.

The picture below shows two 2-component links, namely, the Hopf link and the Whitehead link. Depending on the orientation of individual components, the Hopf link has linking number ± 1; the Whitehead link has linking number 0, irrespective of the orientation.

REMARK 2.6.6. A knot and its null-homologous longitude have linking number 0.

The following exercise illustrates the care one has to take while identifying a null homologous longitude.

EXERCISE 2.6.7. In Figure 2.21[12] find the linking number of the two parallel curves on the left. Do the same for the curves on the right. The parallel curve in the left figure is said to have the *blackboard framing.*

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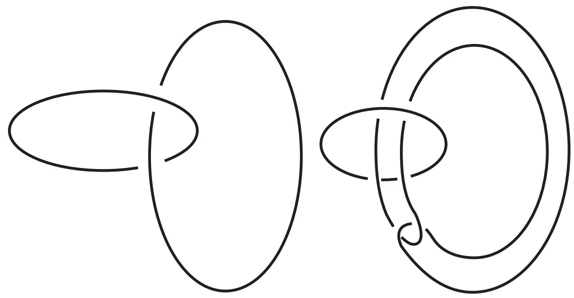

FIGURE 2.20. Hopf link and Whitehead link

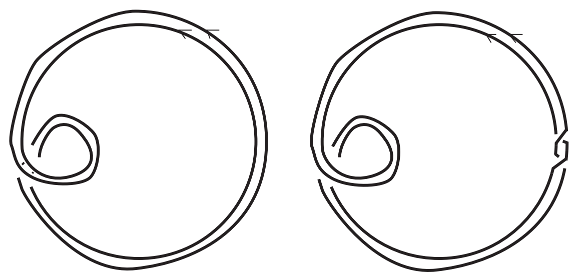

FIGURE 2.21.

EXERCISE 2.6.8. Find the linking number between the right handed trefoil shown in Figure 2.12 and a parallel curve with the blackboard framing oriented in the same manner as the knot itself. Draw a null homologous longitude to the right handed trefoil.

2.6.9. **Knot Surgery**

Let K be a knot in S^3, and let the meridian m and longitude l of K be as in Theorem 2.6.2. Remember that the longitude will have linking number 0 with the knot. Let $V = S^1 \times B^2$ with meridian $\mu = \{(1,0)\} \times \partial B^2$ and longitude $\lambda = S^1 \times \{(1,0)\}$. Let p, $q \in \mathbb{Z}$ be relatively prime integers.

The $\frac{p}{q}$ surgery on S^3 along the knot K is the closed manifold $K_{p/q}$ obtained as the boundary union $X(K) \cup V$ such that the attaching map h takes the meridian μ of V to $pm + ql$ and the longitude λ to $am + bl$, where $a, b \in \mathbb{Z}$. As in Example 2.5.4, we necessarily have $pb - aq = 1$, since h being a diffeomorphism of $S^1 \times S^1$ to itself induces an isomorphism on $\mathbb{Z} \times \mathbb{Z} \cong \pi_1(S^1 \times S^1)$.

Consider the Mayer-Vietoris sequence below.

$$\cdots \to H_1(S^1 \times S^1) \xrightarrow{\phi} H_1(X(K)) \oplus H_1(V) \to H_1(K_{p/q}) \to 0$$

Note that $H_1(X(K))$ is generated by m, $H_1(V)$ is generated by λ, and the homomorphism ϕ sends μ to $pm \oplus 0\lambda$ and λ to $am \oplus \lambda$. It follows that $H_1(K_{p/q}) \cong \mathbb{Z}/p\mathbb{Z}$.

The 0-surgery along a knot is the manifold obtained by gluing a solid torus to the knot exterior in such a way that the meridian of the solid torus is identified with a null-homologous longitude to the knot.

Exercise 2.6.10. Show that the 0-surgery on an unknot in S^3 is a manifold homeomorphic to $S^1 \times S^2$. Exercise 2.1.1 (ii) may be helpful.

Exercise 2.6.11. Show that the 0-surgery $M(K, 0)$ on S^3 along a knot K has the homology of $S^1 \times S^2$.

Exercise 2.6.12. The lens[13] space $L(p, q)$ is the $\frac{p}{q}$ surgery on the unknot. Show that $H_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}$.

See [**?**, Chapter 9B] for more on lens spaces.

2.7. Seifert surfaces

Every knot bounds an orientable surface embedded in S^3. Such a surface is called a Seifert surface for the knot. The surface itself is not unique, but it leads to many knot invariants which we describe in Section 2.10. We begin with an existence proof that relies on obstruction theory, smooth approximation, and transversality.

Let's begin by reviewing some facts from obstruction theory and transversality that we need to use both here and in Section 4.1.

2.7.1. **Obstruction Theory**

A smooth manifold X (with or without boundary) admits the structure of a CW complex, with the boundary as a subcomplex. In the proof of the existence of a Seifert surface in 2.7.2 below, we will be interested in extending a smooth map $f\colon A \to S^1$ defined on a subcomplex A containing the boundary of X, to the entire manifold X.

A continuous extension of f is best achieved by processing it one skeleton at a time. The map f is first defined arbitrarily on 0-cells of $X - A$, and since S^1 is path-connected, it is naturally extended to the 1-skeleton. Denoting the n-skeleton of X by X_n, suppose that $f\colon A \cup X_n \to S^1$ is defined. The obstruction to extending it to X_{n+1} lies in

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$H^{n+1}(X, A; \pi_n(S^1))$. As $\pi_n(S^1) = 0$, for $n > 1$, the only possibly non-trivial case is extension to the 2-skeleton. Moreover, if a map is defined on an n-skeleton of a relative CW-complex, and takes values in a contractible space, then there is no obstruction to extending it to the $n+1$ skeleton. See [**?**] for more on CW complexes and [**?**] for details of the obstruction theory argument.

Once f is defined on all of X, by the Smooth Approximation Theorem [**?**, Chapter II, Theorems 11.7 and 11.8], a small perturbation, that leaves both $f_{|A}$ and the homotopy class of f unchanged, will ensure smoothness. Next, by transversality [**?**], almost every point of S^1 is a regular value of the smooth maps f and $f_{|\partial X}$, and the pre-image under f of a regular value is a submanifold of X of dimension equal to $\dim(X) - 1$.

The manifold X of interest to us in this section is the knot complement $S^3 - K$ or the compact knot exterior (complement of an open tubular neighborhood) $X(K)$ in the three-sphere.

THEOREM 2.7.2. *A knot $K \subset S^3$ bounds an orientable surface embedded in S^3. Moreover, given any orientable surface $F \subset S^3$, such that $K = \partial F$, after a small perturbation if necessary, there is a map $f\colon\ S^3 - K \to S^1$ transverse to $1 \in S^1$ with $f^{-1}(1) = \mathrm{Int}(F)$.*

Proof We will begin with a proof of the second statement. Let K be a knot in S^3, and F an orientable surface bounded by K. Choose a tubular neighborhood $N(K) \cong S^1 \times B^2$ such that its intersection with F is a neighborhood of K in F, and let $m \subset \partial N(K)$ be a meridian, the class of which generates $H_1(S^3 - K)$. Now, let $l = \partial N(K) \cap F$; this is a longitude of K that is null-homologous in $S^3 - K$.

Let $S^1 = \{\, e^{i\alpha} :\ 0 \le \alpha < 2\pi \,\}$ and $B^2 = \{\, te^{i\theta} :\ |t| \le 1,\, 0 \le \theta < 2\pi \,\}$. We choose $S^1 \times B^2 \xrightarrow{\phi} N(K)$ so that $F \cap N(K) = \phi\{\, (e^{i\alpha}, te^{i\theta}) \,|\, \theta = 0 \text{ and } 0 \le t \le 1 \,\} \cong S^1 \times [0,1]$. The meridian m is $\phi(1 \times \partial B^2)$, and we have K and l correspond to $S^1 \times 0$ and $S^1 \times 1$, respectively. Define a function $f\colon N(K) - K \cong S^1 \times (B^2 - 0) \to S^1$ as $\phi(z, w) \xrightarrow{f} \frac{w}{|w|}$. Clearly f takes the constant value 1 on $N(K) \cap \mathrm{Int}(F)$; define f to equal 1 on $F - N(K)$.

Extending f to $S^3 - K$ is achieved in two steps. We first extend f to a collar neighborhood $C \cong (\mathrm{Int}(F) \times [-1, 1])$ of $\mathrm{Int}(F)$ in S^3. Note that $C - \mathrm{Int}(F)$ consists of two components, C_1 and C_2. The function f maps $C_i \cap N(K)$ onto an arc $A_i \subset S^1$, and $A_1 \cap A_2$ is empty. Moreover, neither A_i contains $1 \in S^1$. As A_i is contractible, the map f restricted to $C_i \cap N(K)$ is easily extended to a map $C_i \to A_i$, for $i = 1, 2$. We now have a function $f\colon (C \cup N(K)) - K \to S^1$ such that $f^{-1}\{1\} = \mathrm{Int}(F)$, and we extend $f_{|\partial(C \cup N(K))}$ to a function from $S^3 - \mathrm{Int}(C \cup N(K))$ to the contractible space $S^1 - \{1\}$.

Conversely, given a knot K, its tubular neighborhood $N(K)$, a meridian m, and a null homologous longitude l, let $X(K) = S^3 - \mathrm{Int}(N(K))$ be the knot exterior. Identify $N(K)$ with $S^1 \times B^2$ so that K, l and m correspond to $S^1 \times 0$, $S^1 \times 1$ and $1 \times \partial B^2$, respectively, and define $f\colon N(K) - K \to S^1$ as above to map l to $1 \in S^1$ and m homeomorphically onto S^1.

Note that $H_1(\partial X(K)) = \mathbb{Z}\times\mathbb{Z}$, generated by the classes m and l, whereas $H_1(X(K)) = \mathbb{Z}$ is generated by $i_*[m]$ only, where $[m] \in H_1(\partial X(K))$ and i denotes the inclusion of $\partial X(K)$ in $X(K)$); l is null homologous in $X(K)$. As f maps l to a single point $1 \in S^1$, on homology $f_*[l] = 0 \in H_1(S^1)$. It follows that f_* factors through $H_1(X(K))$. That is, $f_* = \psi \circ i_*$, where $\psi\colon\ H_1(X(K)) \to H_1(S^1)$. Since S^1 is $K(\mathbb{Z}, 1)$, an Eilenberg-Maclane space[14], there is a map $\tilde{f}\colon\ X(K) \to S^1$ such that $\tilde{f}_* = \psi$ and $\tilde{f}_{|\partial X(K)}$ is homotopic to f. Throughout this argument, we may assume f and $\tilde{f}$ to be equal on a small neighborhood of the longitude l. It is clear that $\tilde{f}_*$ has to map $i_*[m]$ to a generator of $H_1(S^1)$. A small perturbation will ensure smoothness. Via transversality, there is a dense set of regular values of $\tilde{f}$ in S^1, and with another small perturbation 1 may be assumed to be a regular value. Now $F = \tilde{f}^{-1}(1)$ is an oriented embedded surface in S^3. □

Note that $H^1(X(K); \mathbb{Z}) \cong \mathrm{Hom}(H_1(X(K); \mathbb{Z})$. So any $c \in H^1(X(K); \mathbb{Z})$ determines a map $\chi\colon\ H_1(X(K)) \to \mathbb{Z}$. In the case that c is the natural generator of $H^1(X(K); \mathbb{Z}) \cong \mathbb{Z}$, we have $\chi(m) = 1$ and $\chi(l) = 0$. In general, for any closed curve $\alpha \subset X(K)$ representing a homology class, we have $\chi(\alpha) = \alpha \cap F$, where F is a Seifert surface, and the intersection is the signed transverse intersection number.

Now we describe Seifert's algorithm, which is a constructive argument based on a knot diagram to obtain a Seifert surface bounded by a knot in S^3.

2.7.3. **Seifert's algorithm**

Begin with an oriented knot diagram, choose any arc and follow it in the direction of the knot orientation. At each crossing, continue the path on the other arc at the crossing in the direction of the orientation. Eventually the path will return to the original point completing a circle, which is now called a Seifert circle[15]. Repeat the process starting with an arc that is not a part of a Seifert circle until no such arcs are left. Now let each

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of the circles bound a disk. Figure 2.22 shows four Seifert circles each bounding a disk. Note that the biggest of the three lighter disks is underneath the dark disk in the middle, not intersecting it.

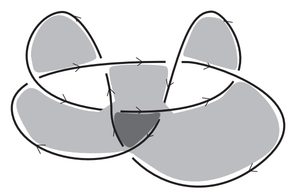

FIGURE 2.22. Seifert circles bounding disks

Finally, we join the disks with half-twisted bands according to the crossings between them. The result is a two-sided (orientable) surface as the orientation on the boundary knot induces a consistent orientation on the surface. See Figure 2.23. We adopt the convention that the oriented knot goes counter clockwise around a positive normal direction to the surface.

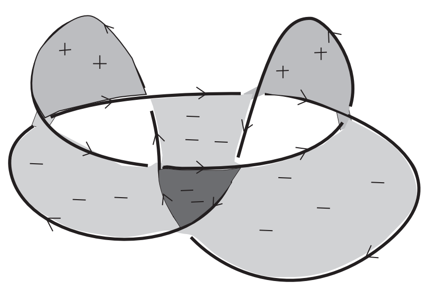

FIGURE 2.23. Seifert surface and orientation

A standard reference for compact surfaces, genus, and Euler characteristic is provided by [**?**, Chapter I]. [16]

EXERCISE 2.7.4. Let K be a knot and D be a diagram for K. Construct a Seifert surface F following Seifert's algorithm. If this algorithm above results in n Seifert circles and m crossings then explain why the Euler characteristic of F is $\chi(F) = n - m$ and why the genus of F is $g(F) = \frac{1-n+m}{2}$.

EXERCISE 2.7.5. For the knot shown in Figure 2.24, use Seifert's algorithm to find a Seifert surface. What is the genus of this surface?

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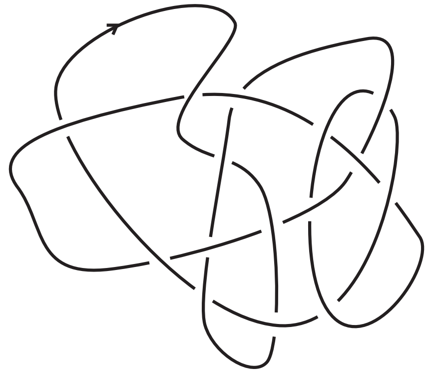

FIGURE 2.24.

DEFINITION 2.7.6. The genus $g(K)$ of a knot K is defined as the minimum among $g(F)$ where F is a compact, orientable surface with $\partial F = K$.

THEOREM 2.7.7. The genus of a nontrivial knot is positive. If K_1, K_2 are knots in S^3, we have $g(K_1 \# K_2) = g(K_1) + g(K_2)$.

The first statement is obvious; if a knot bounds an embedded 2-disk in S^3, it is the unknot (see 2F7 in [**?**]). One can form connected sums of Seifert surfaces for K_1 and K_2, respectively, to obtain a Seifert surface for $K_1 \# K_2$, and therefore it is clear that $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$. The reverse inequality is a bit more work and involves an"inner-most circle argument." For details, see Proposition 7.4 in [**?**].

EXERCISE 2.7.8. Draw genus one Seifert surfaces for the trefoil and the Figure 8 knot.

Note that a boundary parallel curve on a Seifert surface for a knot K is a null-homologous longitude for K, as defined in Theorem 2.6.2 and Remark 2.6.6.

2.8. Seifert forms and matrices

DEFINITION 2.8.1. The Seifert form θ, also known as Seifert pairing, associated to the Seifert surface F is the bilinear map $H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Z}$ defined by $\theta(x,y) = \mathrm{l}k(x, i_+(y))$, where lk denotes the linking number in S_3, and i_+ is the map induced on homology by the positive push-off $F \to (S_3 - F)$.

Note that $H_1(F;\mathbb{Z})$ is a free abelian group of rank $2g$, where g is the genus of the Seifert surface. If we fix a basis $\{x_1, x_2, \dots, x_{2g}\}$ of $H_1(F)$, the matrix

$$A \;=\; \big(\,\theta(x_i, x_j)\,\big)_{i,j}$$

is called a Seifert matrix.

EXERCISE 2.8.2. For the Pretzel knot $P(p, q, r)$ seen in Figure 2.25, find the Seifert matrix with respect to the basis $\{x, y\}$ of $H_1(F)$. Note that p, q, r indicate half-twists.

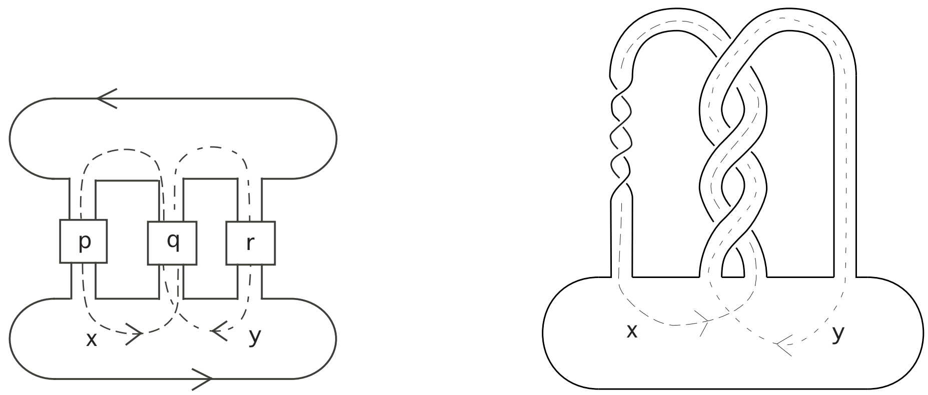

FIGURE 2.25. General Pretzel knot $P(p, q, r)$ and a "disk with bands" representation of $P(1, 3, -3)$

See Proposition 8.2 in Chapter 8 B of [?] for the details that any knot may be represented as the boundary of an "oriented disk with bands" as seen on the right side in Figure 2.25 above. As a consequence, a basis of $H_1(F)$ represented by simple closed curves can be found.

EXERCISE 2.8.3. Show that with an appropriate choice of basis for $H_1(F)$ the Seifert matrix for the left handed trefoil (mirror image of the one in Figure 2.12) is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

EXERCISE 2.8.4. Find a Seifert matrix for the right handed trefoil in Figure 2.12.

EXERCISE 2.8.5. Find curves representing generators of H_1 on a Seifert surface for the Figure 8 knot 4_1. Find the Seifert matrix with respect to this basis.

EXERCISE 2.8.6. Find a Seifert surface and a Seifert matrix for Stevedore's knot 6_1.

EXERCISE 2.8.7. If K bounds a Seifert surface F, then there are natural Seifert surfaces bounded by mK, rK, and $-K = mrK$. Relate the Seifert forms and Seifert matrices for each, being careful about choice of bases. Hint: If A is a Seifert matrix for K, you will see that $-A$, A^T and $-A^T$ occur; which one corresponds to which knot?

EXERCISE 2.8.8. [17]Let G be a free abelian group and G^* denote $\mathrm{Hom}(G;\mathbb{Z})$, the group of homomorphisms from G to $\mathbb{Z}$. Given a bilinear pairing $W\colon G \times G \to \mathbb{Z}$, there is a homomorphism $w\colon G \to G^*$, given by $w(x)(y) = W(x,y)$. The dual homomorphism $w^*\colon G^{**} \to G^*$ is charaterized by $w^*(f)(g) = f(w(g))$, for $f \in G^{**}$ and $g \in G$. Using the canonical isomorphism $G \cong G^{**}$ we have $w^*\colon G \to G^*$. If a basis for G is chosen, the matrix representatives of w and w^* are the transpose of each other. Applying this to the Seifert form and by abuse of notation denoting both W and w by θ, show that $\theta - \theta^*$ represents the intersection form of F; that is,

$$(\theta - \theta^*)(x)(y) = x \cap y,$$

which is the oriented intersection number of classes in $H_1(F)$.

It follows that $\theta - \theta^*$ is skew symmetric and unimodular. A *symplectic basis* for $H_1(F)$ is a basis for which the intersection form has matrix representation a direct sum of the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We can conclude the following:

THEOREM 2.8.9. Let A be a Seifert matrix and let A^T denote the transpose of A. Then $(A - A^T)$ is skew symmetric and has determinant 1. Working modulo 2, it follows that $A + A^T$ has odd determinant.

Homological interpretations

One has $H_1(F) \cong H_1(S^3 - F)$. One way to see this is to consider a deformation retraction of F to its 1–skeleton. Alternatively, one can apply Alexander duality.

Linking number defines a pairing, $H_1(F) \times H_1(S^3 - F) \to \mathbb{Z}$. By considering the 1–skeleton of F, the pairing is seen to be nonsingular; thus it gives an isomorphism $g\colon H_1(S^3 - F) \to H^1(F)$. Via Poincare duality (or the intersection pairing on $H_1(F)$), there is an isomorphism $p\colon H^1(F) \to H_1(F)$. Thus, there is a composition

$$p \circ g \circ i_+\colon H_1(F) \to H_1(F),$$

where i_+ is as in Defintion 2.8.1.

EXERCISE 2.8.10. Given a basis for $H_1(F)$, express the map $p \circ g \circ i_+\colon H_1(F) \to H_1(F)$ in terms of the Seifert matrix.

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Exercise 2.8.11. If A is an integer matrix satisfying $\det(A - A^T) = 1$, show that there is a knot K having Seifert matrix A.

To prove this, first show that by a change of basis, $A - A^T$ is a direct sum of matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then build a Seifert surface as a disk with bands added, similar to the second illustration in Figure 2.25.

In light of the above exercise, we will call any integer matrix A satisfying $\det(A-A^T) = 1$, a Seifert matrix or an abstract Seifert matrix. Note that such a matrix is necessarily even dimensional.

2.9. S-equivalence

There are many choices involved in the definition of a Seifert matrix. A knot can bound distinct Seifert surfaces, or different bases for $H_1(F)$ may be chosen. In order to define a knot invariant using a Seifert matrix, we need to consider an equivalence class of Seifert matrices which allows for such modifications and the invariants should be well-defined on such a class. In this section, we discuss these modifications.

Suppose that a knot $K \subset S^3$ bounds a Seifert surface F. Let α be an embedded arc in S^3 meeting F only at its endpoints, away from K. Then F can be modified by removing small disks around $\alpha \cap F$ and attaching an annulus, $S^1 \times I$, on the boundary of a tubular neighborhood of α. This is one form of stabilization. A simpler stabilization consists of adding to F a disjoint embedded 2–sphere. Seifert surfaces are called stably equivalent, if a succession of such stabilizations and corresponding destabilizations converts one to a surface isotopic to the other.

Theorem 2.9.1. If F_1 and F_2 are Seifert surfaces for K, then they are stably equivalent.

Proof By Theorem 2.7.2, there are functions $g_i \colon X(K) \to S^1$ such that $F_i = g_i^{-1}(p)$ for some regular value p of g_i. The functions correspond to the same element of $H^1(X(K)) \cong \mathrm{Hom}(H_1(X(K);\mathbb{Z}))$, and thus are homotopic. Let G be such a homotopy that is transverse to p. Then $G\colon\ X(K) \times I \to S^1$ with $g_i(x) = G(x,i)$.

Let $g_t = G|_{X(K)\times t}$. After performing a homotopy, we can assume that for almost all t, $g_t^{-1}(p)$ is an embedded surface. The critical points correspond to stabilizations or destabilizations of the surface.[18] $\square$

EXERCISE 2.9.2. If F_1 and F_2 are connected Seifert surfaces for K, show that they are stably equivalent via a series of connected Seifert surfaces.

OPEN QUESTION 2.9.3. A Seifert surface F for K is called *free* if $S^3 - \mathring{N}(F)$ is a solid handlebody. Seifert's algorithm for building Seifert surfaces yields free Seifert surfaces. If F_1 and F_2 are free Seifert surfaces for K, are they stably equivalent via a sequence of free Seifert surfaces?

EXERCISE 2.9.4. Suppose that F is a disk with bands attached. Consider the special case of adding genus to the surface by removing two disks and adding a tube which does not link any of the existing generators. Observe that this would change the $n \times n$ matrix A to an $(n+2) \times (n+2)$ matrix which is the diagonal sum $A \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

In general, if stabilization increases the genus of the Seifert surface by 1, then two new generators are added to $H_1(F)$. These can be selected so that the effect on an associated Seifert matrix A is to add two new rows and columns, changing A into the matrix:

$$\begin{pmatrix} & & & 0 & a_1 \\ & A & & \vdots & \vdots \\ & & & 0 & a_{2g} \\ 0 & \dots & 0 & 0 & 0 \\ a_1 & \dots & a_{2g} & 1 & 0 \end{pmatrix}.$$

Using a different basis for $H_1(F)$ changes the Seifert matrix by congruence. That is, we will have PAP^T instead of A where P is the matrix that gives the basis change and P^T denotes the transpose of P.

Two Seifert matrices are called S–equivalent, if they differ by a sequence of stabilizations and congruences.

THEOREM 2.9.5. Seifert matrices for isotopic knots are S-equivalent [**?**].

For more on S-equivalence see [**?**, §9].

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REMARK 2.9.6. It was shown in [**?**] that two (oriented) knots have S-equivalent matrices if and only if they have diagrams which are related by the following diagram move known as the double-delta or the d-delta move.

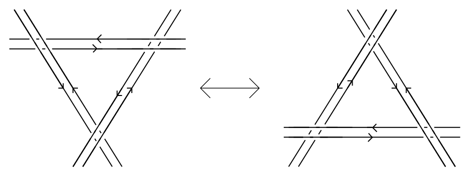

FIGURE 2.26. The d-delta move

2.10. Knot invariants arising from Seifert matrices

From Theorem 2.9.5, it follows that any object defined in terms of a Seifert matrix is an invariant of the knot if and only if it remains unchanged under S-equivalence. Invariants which are determined by a Seifert matrix are sometimes known as algebraic or abelian invariants. If two knots have S-equivalent Seifert matrices, their abelian invariants will match[19]. The reader can check that the following are invariants of the S–equivalence class and therefore, although defined in terms of a Seifert matrix, they yield invariants of knots. Standard references for proofs are [**?, ?**].

For the rest of this section, let K be a knot in S^3 and let A be a Seifert matrix for K.

2.10.1. **Determinant**

The determinant of a knot is $\det(K) = |\det(A + A^T)|$.
From Theorem 2.8.9, it follows that $\det(K)$ is always odd.[20]

2.10.2. **Alexander polynomial**

The Alexander polynomial for knot K is $\Delta_K(t) = \Delta_A(t) = \det(A - tA^T)$. We will drop the subscripts when there is no confusion.

EXERCISE 2.10.3. Using properties of Seifert surfaces established earlier in this chapter, show that

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- $\Delta(1) = 1$, and
- $\Delta(t) = t^{\dim(A)}\Delta(t^{-1})$.

The Alexander polynomial is a well defined knot invariant up to multiples of $\pm t^n$. That is, if two different Seifert surfaces for knot K are used to compute the Alexander polynomial, then one polynomial will be $\pm t^n$ times the other, for some power n. Since $\Delta_K(-1) = \det(K)$, keeping track of signs gives no additional information. The polynomial is usually defined as a Laurent polynomial, i.e. an element in $\mathbb{Z}[t, t^{-1}]$, well defined up to multiplication by units in $\mathbb{Z}[t, t^{-1}]$.

When $\Delta_1(t) = \pm t^k \Delta_2(t)$ for some integer value of k, we use the notation $\Delta_1 \doteq \Delta_2$. Along with the symmetry $\Delta(t^{-1}) \doteq \Delta(t)$, unique factorization in $\mathbb{Z}[t, t^{-1}]$ implies that the Alexander polynomial factors as a product

$$\Delta \doteq p_1^{\epsilon_1} p_2^{\epsilon_2} \dots p_m^{\epsilon_m} (q_1 q_1')^{\delta_1} (q_2 q_2')^{\delta_2} \dots (q_n q_n')^{\delta_n}$$

where the p_i, q_i and q_i' are distinct irreducible polynomials, each of which is equal to ± 1[21] at $t = 1$, with the p_i symmetric and $q_i'(t^{-1}) \doteq q_i(t)$ for all i.

Exercise 2.10.4. Compute the Alexander polynomial of the pretzel knots illustrated in Figure 2.25.

Theorem 2.10.5. For a knot $K \subset S^3$, we have $\Delta_K \doteq \Delta_{mK} \doteq \Delta_{rK}$ and $\deg(\Delta_K) \leq 2g(K)$. Moreover, for knots $K_1,\ K_2 \in S^3$, we have $\Delta_{K_1 \# K_2} = \Delta_{K_1} \cdot \Delta_{K_2}$.

The first part of the theorem is easy to prove. The second statement regarding connected sums is Proposition 8.14 in [?]. The proof consists of showing that a Seifert surface for the connected sum is obtained by taking the connected sum of Seifert surfaces of individual knots and the corresponding Seifert matrix is the orthogonal sum of the individual Seifert matrices.

Exercise 2.10.6. Given a polynomial $\Delta(t)$ that satisfies the conditions of 2.10.3, show that there is an abstract Seifert matrix A as in 2.8.11 with $\det(A - tA^T) = \Delta(t)$.

Remark 2.10.7. A polynomial Δ satisfying the conditions in 2.10.3 is called an (abstract) Alexander polynomial. It follows from 2.8.11 and 2.10.6 that there is a knot K in S^3 such that $\Delta = \Delta_K$.

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See [?] for a surgery-theoretic proof that given an abstract Alexander polynomial, there is a knot with that as its Alexander polynomial.

2.10.8. **The Arf Invariant:**

The Arf invariant defines a $\mathbb{Z}/2\mathbb{Z}$ homomorphism on the Witt group of $\mathbb{Z}/2\mathbb{Z}$ quadratic forms. [22](We will use the notation $\mathbb{Z}/p\mathbb{Z}$ for the finite cyclic group of order p; we will reserve $\mathbb{Z}_p$ for p-adic integers.)

Given a $(2g) \times (2g)$ Seifert matrix A, one defines a $\mathbb{Z}/2\mathbb{Z}$–valued quadratic form on the vector space $(\mathbb{Z}/2\mathbb{Z})^{2g}$ by $q(x) = xAx^{\tau} \bmod 2$. This is a nonsingular quadratic form in the sense that $q(x+y) - q(x) - q(y) = x \cdot y$, where the nonsingular bilinear pairing $x \cdot y$ is given by the matrix $A + A^T$. (Recall that the determinant of $A + A^T$ is odd.)

The simplest definition is that $\mathrm{Arf}(q) = 0$ or $\mathrm{Arf}(q) = 1$ depending on whether q takes value 0 or 1, respectively, on a majority of elements in the vector space. See [?] or [?].

Equivalently: There is a symplectic basis x_i, y_i, $1 \leq i \leq g$, of $(\mathbb{Z}/2\mathbb{Z})^{2g}$ such that $x_i \cdot y_j = \delta_{i,j}$ and $(x_i, x_j) = (y_i, y_j) = 0$. With respect to such a basis, and corresponding Seifert matrix $A = (a_{i,j})$, the Arf invariant is

$$\mathrm{Arf}(A) \equiv \sum_{i=1}^{g} a_{2i-1,2i}\, a_{2i,2i} \bmod 2.$$

This invariant was first defined by Robertello in [?]. Murasugi [?, Theorem 2] observed that $\mathrm{Arf}(A) = 0$ if and only if $\Delta_A(-1) = \pm 1 \mod 8$.

2.10.9. **Signatures**

By applying the Jacobi-Sylvester Inertia Theorem [?, III.2.5] to the nonsingular, symmetric, integral matrix $A + A^T$, we know that the number of positive entries minus the number of negative entries in a diagonalization of $A + A^T$ over the reals is an invariant of the matrix. See [?, Chapter 2]. It turns out that this difference, in fact, is invariant under S-equivalence [?].

Definition 2.10.10. The signature $\sigma(K)$ of a knot K is the number of positive entries minus the number of negative entries in a diagonalization of $A + A^T$, where A is a Seifert matrix for the knot.

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THEOREM 2.10.11. The signature of a knot has the following properties:

- $\sigma(K)$ is an even number for any knot K,
- $\sigma(rK) = \sigma(K)$,
- $\sigma(mK) = -\sigma(K)$,
- $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$.

EXERCISE 2.10.12. Compute the signature of the left handed trefoil and of the Figure 8 knot. You may refer to exercises 2.8.4 and 2.8.5.

As two Hermitian forms over a field with involution (eg. $\mathbb{C}$) are isomorphic if and only if their underlying quadratic spaces are isomorphic over the fixed field (eg. $\mathbb{R}$), signatures are also well-defined for Hermitian forms. (See Appendix 2 of [**?**].) This leads to the following definition.

DEFINITION 2.10.13. For a unit norm complex number $\omega = e^{i\theta}$, the ω-signature σ^*_ω or σ_θ of a knot is the number of positive entries minus the number of negative entries in a diagonalization of the Hermitian matrix $(1-\omega)A+(1-\bar{\omega})A^T$, where A is a Seifert matrix for the knot.

Clearly, σ^*_{-1} or σ_π is the standard signature σ.

EXERCISE 2.10.14. Verify the following statements (See [**?**]):

(1) The ω–signature defines an integer valued function on the unit circle, with a finite set of discontinuities. Away from the discontinuities, it is even valued. The discontinuities occur only at roots of the Alexander polynomial.
(2) The determinant and Alexander polynomial are multiplicative under connected sum and the ω–signatures are additive. (See [**?**].)

EXERCISE 2.10.15. Suppose that a knot K bounds a genus one Seifert surface. Explain how to determine the signature function (up to sign) and Alexander polynomial from the determinant of K.

DEFINITION 2.10.16. The Tristram-Levine signature σ_ω of a knot K at a unit complex number ω is defined to be $\lim(\sigma^*_{\omega^+}(K) + \sigma^*_{\omega^-}(K))/2$ where ω^+ and ω^- are unit complex numbers approaching ω with arguments above and below that of ω.

The Tristram-Levine signatures of a knot K define a function on the unit circle. These functions are knot invariants. (See Therem 4.3.9 in Chapter 4.) It is easy to establish that they are additive under connected sums of knots; that is:

$$\sigma_\omega(K_1 \# K_2) = \sigma_\omega(K_1) + \sigma_\omega(K_2).$$

They are unchanged by reversal of string orientation; that is:

$$\sigma_\omega(K) = \sigma_\omega(rK)$$

and are negatives of each other for mirror images:

$$\sigma_\omega(K) = -\sigma_\omega(mK).$$

See 5.2.1 and 5.3.1 for more on the signature functions. Also, see [**?**, **?**] or [**?**] for a general survey of signature invariants.

CHAPTER 3

Knot concordance

In 1926 Artin [?] described the construction of certain smooth, knotted 2–spheres in $\mathbb{R}^4$. The intersection of each of these with the standard $\mathbb{R}^3 \subset \mathbb{R}^4$ was a nontrivial knot in $\mathbb{R}^3$ which bounds a smooth disk in the upper half-space. This was the beginning of the study of knot concordance. Initially it seemed possible that every knot can occur as such a slice of a knotted 2–sphere, and it wasn't until the early 1960s that Murasugi [?] and Fox and Milnor [?, ?] succeeded at proving that some knots are not slice.

It is worth noting that if a knot bounds an embedded 2-disk in S^3, it is the unknot (see 2F7 in [?]). On the other hand, for any knot, coning (S^3, K) yields a topological 2-disk in B^4 bounded by the knot. Such a disk is not smooth at the cone point if K is nontrivial in S^3.

Definition 3.0.1. A knot K is called slice if there is a pair (B^4, D^2) such that $(S^3, K) = \partial(B^4, D^2)$, where B^4 is the 4–ball and D^2 is a smoothly and properly embedded 2–disk which we will call a slice disk for K. (Superscripts indicate dimension and we may occasionally drop them.)

This defines a smoothly slice knot. There is a related topological notion of sliceness, which requires the disk embedding to be topologically locally flat.

Note that an m-dimensional topological manifold M topologically embedded in an n-dimensional topological manifold N, $n > m$, is considered locally flat, if at each point of M, there is a neighborhood $U \subset N$, such that the pair $(U, U \cap M)$ is diffeomorphic to the standard disk pair (B^n, B^m). A knot K in S^3 is said to be topologically slice if it bounds a locally flat 2-disk D properly embedded in the 4-ball B.

Freedman and Quinn [?] showed that a locally flat surface in a topological four-manifold has an embedded normal bundle and consequently, a tubular neighborhood. Conversely, existence of a tubular neighborhood guarantees that the embedding is locally flat. So an equivalent condition for K to be topologically slice is that there is a (topological) 2-disk $D \subset B^4$ with a tubular neighborhood $N \cong D \times B^2$ such that $N \cap S^3$ is a tubular neighborhood of the knot K. As a smooth embedding has a tubular neighborhood given by a normal bundle, it follows that a smoothly slice knot is topologically slice, however, the converse is not true. In [**?**] Freedman showed that any knot with Alexander polynomial one is topologically slice. Since then, using various techniques, several mathematicians have obtained examples of Alexander polynomial one knots which are not smoothly slice, which shows that smooth sliceness is a stronger condition. See [**?**, **?**] for early examples which use Donaldson's work from [**?**]. Techniques from contact geometry are used in [**?**, **?**, **?**]. Examples of other topologically slice polynomials[1] and corresponding non-smoothly slice knots are given in [**?**, **?**].

Unless mentioned otherwise, by "slice" we will mean smoothly slice.

3.1. Slices and ribbons

We now focus on smoothly slice knots and review the handle-body description of the slice disk as in Section 2.2.6 and Figure 2.7.

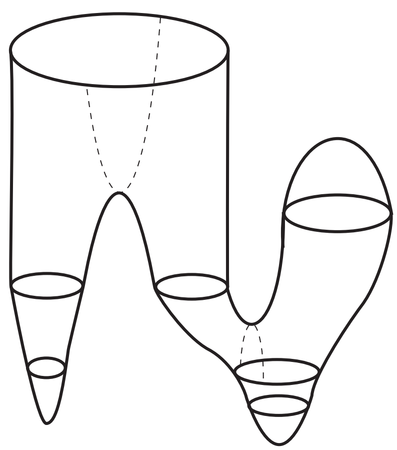

FIGURE 3.1. Prototype for a smooth slice disk

Let $D \subset B^4$ be a slice disk for a knot K. The standard norm in $\mathbb{R}^4$ gives the radius function on B^4, which restricts to give a function h on D, and after a small perturbation of D, h can be assumed to be a Morse function. As mentioned in Section 2.2, further perturbation assures that the critical points occur in the order of their index as shown in the prototype below which is described in detail in 2.2.6. [2]

1

2

Supposing that the slice disk for a knot misses the origin in B^4, it can be described via a series of diagrams of the cross-sections $K_t = S^3_t \cap D$; here S^3_t is the sphere of radius t. Generically, there is a finite set of values of t for which the intersection is not a link, and between any of these two critical values D is a product. Arranging the indices of the critical values in order, the cross-sections, beginning at a small radius, consist first of a collection points and then of circles forming an unlink. As the radius is increased, the boundary circles are joined by a series of band moves, ultimately resulting in a link consisting of one circle, possibly knotted, and an unlink far from the first component. Passing the last critical values eliminates the unlink. An Euler characteristic argument shows that the resulting surface is a disk if the number of points in the initial collection (index 0 critical points) plus the number of circles in the final unlink that are capped off (index 2), minus the number of bands (index 1), equals 1.

THEOREM 3.1.1. *For any knot K, $K\# - K$ is a slice knot.*

Before proving Theorem 3.1.1, we consider the example of the square knot, which is a connected sum of the trefoil knot K and its inverse $-K$ (mirror image with reversed string orientation).

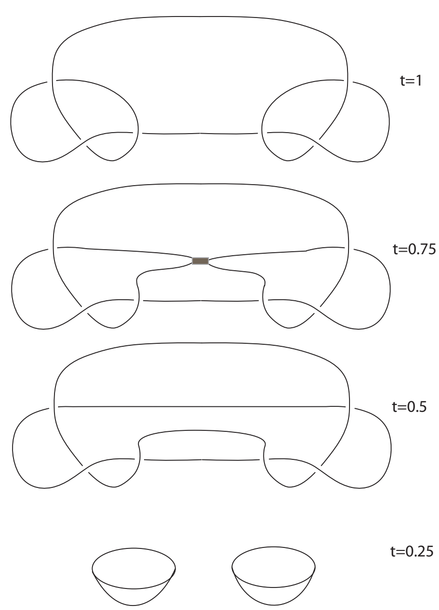

FIGURE 3.2. A slice movie for the square knot

In Figure 3.2 we see cross-sections of a slice disk D for the square knot. As above, by denoting the sphere of radius t by S^3_t, $1 \geq t > 0$, the diagrams at various t values represent $S^3_t \cap D$ and are referred to as "stills" from the slice "movie" with t considered as time. Looking at this top down, we have the knot at $t = 1$, followed by a saddle point (index 1) where the circle pinches together (see the gray band at $t = 0.75$) splitting into two circles at $t = 0.5$. The 2-component link at $t = 0.5$ is isotopic to the two unlinked unknotted circles shown at $t = 0.25$ which bound disjoint disks resulting in local minima (index 0).

Proof of Theorem 3.1.1: [3]

We provide three proofs for this claim. Then first is very diagrammatic immitation of the idea used in Figure 3.2. The second is other more parametric using the construction of higher dimensional spun knots first described by Artin in [**?**]. (Also see Section 3J5 in [**?**].) Finally we give a more topological argument. Each of these illustrates different ideas which recur in the study of knot concordance.

Fix a diagram D for the knot K containing k crossings. Let $-D$ be the diagram for $-K$ given by reflecting over the y-axis and reversing the arrow. $K\# - K$ now has a diagram $D\# - D$ given by adding a band from D to the corresponding point on $-D$. Start at a point on the boundary of this band, and follow the arrow forward (into D) and backwards (into $-D$) until you encounter a crossing. Attach to $K\# - K$ a band as in Figure and push the resulting curve slightly into the interior of the 4-ball. What results is a planar surface cobordism from $K\# - K$ in S^3_1 to a new very symmetric diagram in $S^3_{1-\epsilon}$ with one added component and one fewer crossing. Iterate this until you have eliminated every crossing. A reasonably straightforward bit of book-keeping reveals that each time a band is added, the number of components increases by one and the number of crossings reduces by one. After all crossings have been removed the resulting link is the unlink in $S^3_{1-k\epsilon}$ whose components you can cap with disks. An Euler characteristic argument reveals that this surface is a disk.

Next we give a less diagrammatic argument which results in a parametrization of this disk. The following sets may be viewed as subsets of $\mathbb{R}^5$.

Let S be the "upper half" of the 3-sphere S^3_+ embedded in $\mathbb{R}^5$. That is,

$$S = \left\{ (x_1, x_2, x_3, x_4, 0) \,|\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1, \, x_4 \geq 0 \right\}.$$

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Clearly, $\partial S = \{\,(x_1, x_2, x_3, 0, 0) \mid x_1^2 + x_2^2 + x_3^2 = 1\,\} \cong S^2$. Let A be an arc in S intersecting ∂S transversely, such that when adjoined with an arc on ∂S it gives a knot isotopic to K.

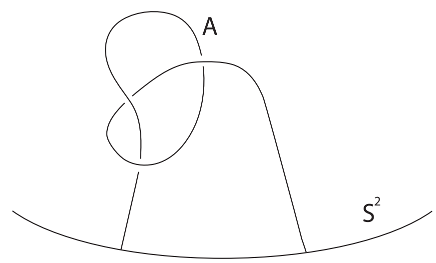

FIGURE 3.3. An arc in the upper half space

For a point $x = (x_1, x_2, x_3, x_4, 0) \in S$ define $x_\theta = (x_1, x_2, x_3, x_4 \cos\theta, x_4 \sin\theta)$. Define the spin X^* of a set $X \subset S$ to be

$$X^* = \{\, x_\theta \mid x \in X, 0 \leq \theta \leq \pi \,\}.$$

Note that $x_4 \sin\theta \geq 0$, for $0 \leq \theta \leq \pi$. It is easy to see that $S_\theta \cong S^4_+ \cong B^4$ and A_θ is a smooth 2-dimensional disk bounded by $K \# -K$.

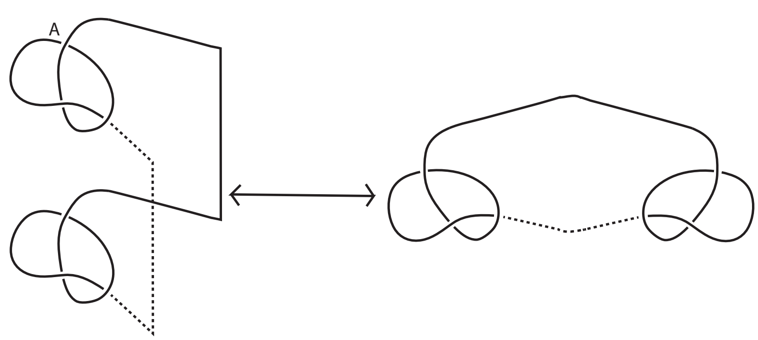

FIGURE 3.4. $\partial(A \times I) \subset S^3 \times I$ and $K \#(-K)$

Alternatively, the fact that $K \# -K$ is slice can be seen as follows. Let B be a small 3-ball in S^3 whose boundary intersects K in two points and the arc in the interior of B is unknotted. The pair $(S^3 - \mathrm{Int}(B), A = K - \mathrm{Int}(B))$ is diffeomorphic to the pair (B^3, J), where J is some knotted arc. The product with the interval $I = [-1, 1]$ gives a pair that is diffeomorphic to (B^4, D), where D is a knotted two disk. Furthermore, recalling the boundary orientation convention as in Section 2.1, we see that ∂D is (up to diffeomorphism) formed from A and $-A$ together with 2 arcs that join the corresponding end points. In Figure 3.4, on the left we see $\partial(A \times I) \subset \partial(B^3 \times I)$ and on the right we see $\partial D^2 = K \# -K \subset S^3$.

□

EXERCISE 3.1.2. Show with a slice movie as in Figure 3.2 that the knot obtained as the connected sum of the Figure 8 knot with itself is slice. (See Exercie 2.3.5.)

Note that the disk in Figure 3.1 has one local maximum or a critical point of index 2, whereas the slice disk in Figure 3.2 for the square knot has no critical points of index 2. The square knot belongs to an easily described class called ribbon knots.

DEFINITION 3.1.3. If a knot bounds a slice disk with no index 2 critical points, it is called a ribbon knot, and such a disk is called a ribbon disk.

EXERCISE 3.1.4. For any knot K, $K\# -K$ is a ribbon knot. Determine which of the three arguments in the proof of 3.1.1 results in a ribbon disk.

Here is a useful 3-dimensional interpretation:

DEFINITION 3.1.5. An immersed 2-disk in S^3 is called a ribbon disk if:

(i) The singularities of the immersion consist only of transverse double points,
(ii) The double point set of the image consists of a collection of arcs, and
(iii) The preimage of each of these arcs is a pair of arcs on the disk, one with endpoints on the boundary of the disk and one contained in the interior of the disk shown to be within the dashed ellipse in Figure 3.5.

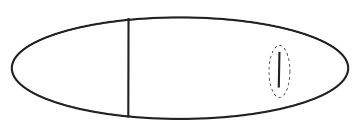

FIGURE 3.5. Preimage of an arc of self-intersection in an immersed ribbon disk in S^3

THEOREM 3.1.6. A knot bounds a properly embedded ribbon disk in B^4 if and only if it is the boundary of an immersed ribbon disk in S^3.

Proof Let's begin with a knot K which is the boundary of a properly embedded ribbon disk D in S^3. That is, assume that the radius function h of B^4 restricted to D is a Morse function, and that $h_{|D}$ does not have index two critical points. Recall from Section 2.2.6 that a critical point of $h_{|D}$ corresponds to a handle with index equal to the Morse index of the critical point. We want to show that there is an immersed ribbon disk in S^3 which is bounded by the knot K. This is achieved as follows.

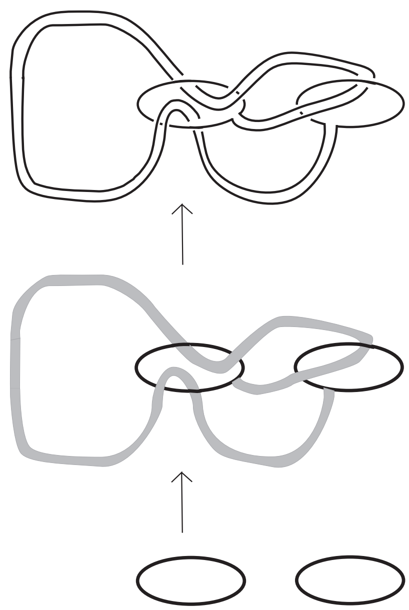

FIGURE 3.6. Constructing a knot from a ribbon disk

Working upwards from the index 0 critical points, which contribute disjoint disks (0-handles) uniquely determined up to isotopy by their number, we see disjoint, unknotted, unlinked circles which are boundaries of the disks. Passing through each index one critical point corresponds to attaching two of these boundary circles together with a band $B^1 \times B^1$ (1-handle) attached along $\partial B^1 \times B^1$, as illustrated in Figure 3.6 above. (Also see the discussion in 2.2.6.) The knot in S^3 is generated by a "band sum" of the circles as seen in the top picture of Figure 3.6. The ribbon immersion in S^3 is obvious.

Conversely, an immersed ribbon in S^3 can be perturbed in B^4, relative its boundary, to form a ribbon disk. Specifically, one would push a small open neighborhood of the interior arc, such as the region marked in Figure 3.5 with the dashed ellipse, inside the interior of the 4-ball to obtain a properly embedded disk in B^4 without index 2 critical points. □

We see below two examples of knots bounding ribbon disks in S^3. Figure 3.1 shows the square knot again and in Figure 3.8 we see a picture of the knot 6_1, also known as Stevedore's knot.

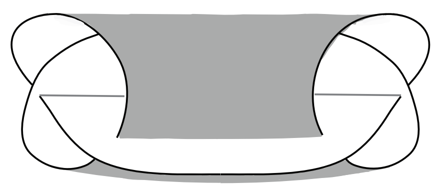

FIGURE 3.7. A Ribbon immersed in S^3 bounded by the Square Knot

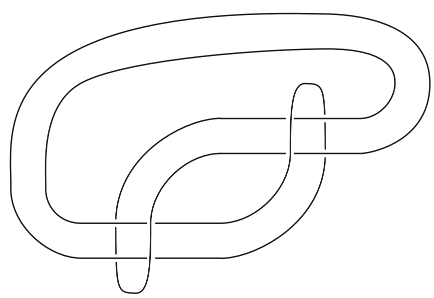

FIGURE 3.8. Stevedore's Knot 6_1

The immersed disk bounded by 6_1 in S^3 is obvious. We see it passing twice through itself.

EXERCISE 3.1.7. Construct a ribbon diagram for the knot 9_{41}[4] similar to the one for Stevedore's knot 6_1.

Theorem 3.1.6 offers an alternative definition of ribbon knots, as knots that bound immersed ribbon disks in S^3. Ribbon knots naturally provide examples of slice knots.

OPEN QUESTION 3.1.8. If K is a slice knot, is K a ribbon knot?

Question 3.1.8 was first posed by Fox in 1966. It appears as Problem 1.33 in [?]. Since ribbon knots are smoothly slice, this question is relevant only in the smooth setting. As mentioned at the beginning of this Section, several examples are known of topologically slice knots which are not smoothly slice, and therefore not ribbon.

[4]Add a figure of 9_{41}

3.1.9. A ribbon disk does not have index two critical points. An Euler characteristic argument shows that a knot is ribbon if and only if, for some n we can make n band or ribbon moves on the knot to obtain an $n+1$ component unlink.

In Figure 3.2 we see one band move on the square knot resulting in a 2-component unlink. In Figure 3.6 we see a ribbon disk in B^4 described going upward from lowest t-value.

3.1.10. A band running from a component to itself will introduce genus if the components, previously split by a band move, are rejoined by another band. In this case we no longer have a disk. For example, see the band moves performed on the trefoil knot in Figure 3.9.

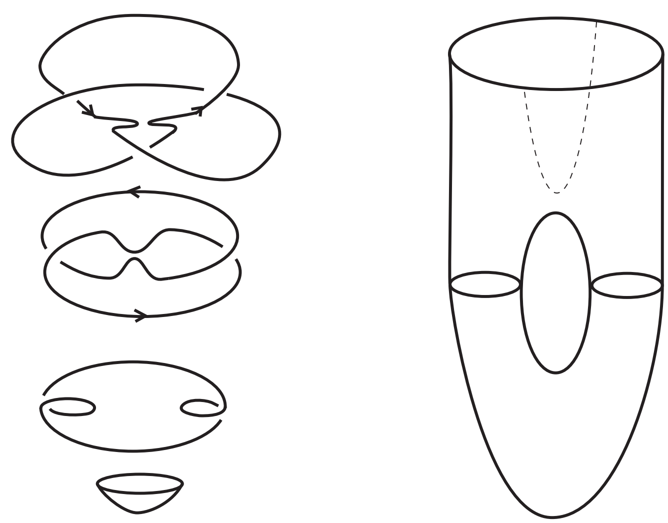

FIGURE 3.9. Band moves on the trefoil giving a genus one surface in the 4-ball.

3.1.11. Note that for any knot in S^3, the interior of a Seifert surface may be pushed inside the 4-ball to obtain a *ribbon surface*, i.e., a surface without index 2 critical points, bounded by the knot. However, in general, such a surface will not be a disk.

Let's briefly discuss the special case of a genus one knot. Suppose that K bounds a Seifert surface F of genus 1 and that there is a homologically nontrivial simple closed curve $\alpha \subset F$. (Requiring that α be homologically nontrivial and simple is equivalent to requiring that it represent a primitive class in $H_1(F)$). Remove the interior of an annular neighborhood of α on F. The resulting bounded surface F' has boundary K and two parallel copies of α, say α_1 and α_2. If these curves bound disjoint disks B_1 and B_2 in B^4, these can be attached to F' to form a disk bounded by K in the 4–ball. One obvious condition needed for the disks to exist is that α represent a slice knot in S^3. The condition that the disks can be made disjoint is equivalent to the condition that $\mathrm{lk}(\alpha_1, \alpha_2) = 0$. Equivalently, $\theta(\alpha, \alpha) = 0$, where θ is the Seifert form associated to F. To understand this linking condition, we have the following:

LEMMA 3.1.12. Suppose that K_1 and K_2 are disjoint knots in S^3, bounding transverse surfaces G_1 and G_2 in B^4. Then the oriented intersection number $\#(G_1 \cap G_2) = \mathrm{lk}(K_1, K_2)$.

Proof First we observe that the intersection number is independent of the choice of surfaces. If surfaces G_1' and G_2' were used, by gluing two copies of B together, we would form closed surfaces G_1'' and G_2'' in S^4. The signed intersection $\#(G_1'' \cap G_2'')$ is equal to the intersection number of the homology classes represented by the surfaces in $H_2(S^4)$. But since this group is trivial, the intersection number must be 0.

We have given an algorithm for computing linking numbers in terms of crossings. Given K_1 and K_2, we can form surfaces bounded by each knot by using the trace of an isotopy of K_1 to a knot that lies entirely above K_2, and then adding disjoint Seifert surfaces for this translated copy of K_1 and K_2. The intersections between these surfaces occur at points corresponding to the crossing changes we made, and these in turn are counted by the linking number. □

REMARK 3.1.13. While the lemma above is stated and proved in the smooth category, the same applies topologically, module an understanding of transverse intersections of locally flat embedded surfaces. See CITE (Freedman-Quinn?)

3.2. The slice disk exterior

Much like the knot exterior in S^3, studying the exterior of a disk in the 4-ball provides useful information. Let K be a slice knot with $(S^3, K) = \partial(B^4, D)$ and let $X(D) = B^4 - \mathrm{Int}(N(D))$, where $N(D)$ denotes a tubular neighborhood of D. In 3.1 we saw a handlebody description of D. We now describe a corresponding handle structure on $X(D)$ as described in 2.2.7. [5]

5

As before, assume that the radius function h on B^4 restricts to a Morse function on D and the critical points of $h_{|D}$ are arranged in increasing order of their Morse index. On intervals between critical values of h, D is a product, and a critical point of Morse index k corresponds to attaching an index k handle $B^k \times B^{2-k}$ to the boundary.

In Figure 3.6 we illustrated the construction of a ribbon disk in B^4. The exterior of this disk is described in Figure 3.10 below. The two dotted circles denote 1-handles $B^1 \times B^3$ each corresponding to the removal of a neighborhood of a disk. The attaching band for the index 1 critical point corresponds to the 2-handle $B^2 \times B^2$ attached along an $S^1 \times B^2$ whose core is the remaining circle and the framing 0 indicates that the meridian of the solid torus is mapped to the meridian of this curve.

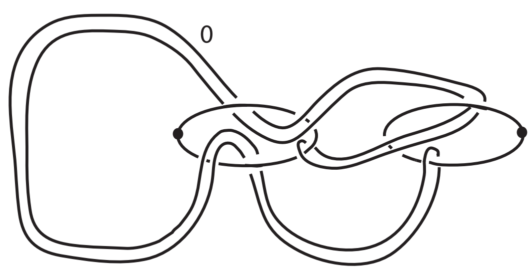

FIGURE 3.10. The exterior of a ribbon disk

After sliding handles and erasing canceling 1- and 2-handles, we obtain Figure 3.11 below. (See Exercise 2.2.8.)

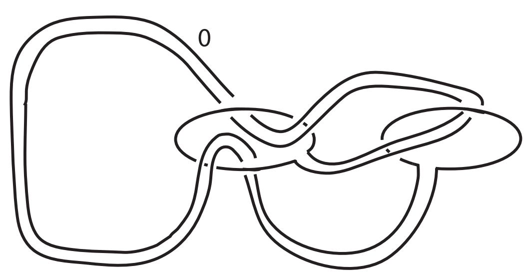

FIGURE 3.11. The exterior of a ribbon disk

A non-ribbon slice disk would also have 2-handles $B^2 \times 0$ corresponding to local maxima of the Morse function given by the radius of the 4-ball restricted to the slice disk. Corresponding to the 2-handles of the disk, 3-handles $B^3 \times B^1$ are attached to the disk exterior.

The following result is now obvious as the handle description of the four manifold provides surgery description for the boundary three-manifold [**?**, Sectiom 5.2]. For the sake of completeness, we outline a proof without using the language of handles. See 2.6.9 for more on 3-manifolds obtained by knot surgery.

LEMMA 3.2.1. Suppose K is a slice knot bounding a slice disk D in B^4. Let $N(D) = D \times B^2$ be a closed tubular neighborhood of D which intersects S^3 in a tubular neighborhood $N(K)$ of K and let $X(D) = B^4 - \mathrm{Int}(N(D))$. Then $\partial X(D)$ is the three-manifold $M(K, 0)$ obtained by the 0-surgery on S^3 along K.

Proof: Note that

$$\partial X(D) \;=\; \big(\partial B^4 - \big(\mathrm{Int}(N(D)) \,\cap\, \partial B^4\,\big)\big) \bigcup \big(\,\partial N(D) \,\cap\, \mathrm{Int}(B^4)\,\big)\,.$$

Since $N(D) \cap \partial B^4 = N(K)$, we have $\partial X(D) \cap \partial B^4 = X(K)$, the knot exterior. Furthermore, $\partial N(D) = (D \times \partial B^2) \bigcup (\partial D \times B^2)$ and therefore $\overline{\partial N(D) \cap \mathrm{Int}(B^4)} = D \times \partial B^2$. Here the bar denotes closure.

It follows that the boundary of the slice disk exterior in B^4 is

$$X(K) \bigcup_{\partial X(K) \cong (\partial D \times \partial B^2)} (D \times \partial B^2),$$

where the identifications along boundary map a meridian of the attached solid torus to some longutude λ of K. In order to see that it must be the nullhomologous longitude, notice that this longitude bounds a pushed off copy of D. Thus K and λ bound disjoint surfaces and so Lemma 3.1.12 implies that $\mathrm{lk}(K, \lambda) = 0$. [6] This is 0-surgery on S^3 along K, as described in 2.6.9. $\square$

Turning a handlebody upside down as in 2.2.3, we see its dual decomposition. In 2.2.7 we saw that an index k critical point of D corresponds to an index $k+1$ handle in $X(D)$. If we begin the construction of $X(D)$ by starting with $X(K) \times I$, a handle previously seen as a $k+1$-handle in $X(D)$ is now seen to have index $4-(k+1)$ in the dual decomposition. Said differently, in a handle decomposition of $X(D)$ relative $X(K)$, we have a j-handle for each critical point of index $3 - j$ on the slice disk D. This perspective is useful while doing the exercise below.

6

EXERCISE 3.2.2. Show that for a ribbon knot K bounding a ribbon disk $D \subset B^4$, the inclusion induced map of $\pi_1(X(K))$ to $\pi_1(X(D))$ is surjective. Hint: Note that the fundamental group is generated by 1-cells.

REMARK 3.2.3. Using the Mayer-Vietoris sequence for the pair $(X(D), N(D))$ in B^4, it is easy to see that $H_1(X(D)) = \mathbb{Z}$ and $H_i(X(D)) = 0$, for all $i > 1$. Furthermore, using the exact homology sequence for the pair $(X(D), \partial X(D))$ and Poincaré-Lefschetz duality, we see that the map $H_1(X(K)) \cong H_1(M(K,0)) \longrightarrow H_1(X(D))$ is an isomorphism. Here $M(K,0) = \partial X(D)$ denotes the 0-surgery on S^3 along K.

At this point, it is worth noting that even though the Ribbon-Slice problem stated in 3.1.8 has interested mathematicians for quite some time, and answered affirmatively for certain infinite families [**?, ?**], there are no good tools to even begin tackling it in its generality. For instance, there is no reason that the conclusion of Exercise 3.2.2 should hold true for slice knots; however, fundamental groups are not easy to compute. Besides, whether or not a knot is ribbon isn't determined by whether or not a *given* slice disk bounded by the knot is a ribbon disk. There may be another which is. So this obstruction cannot be used effectively to find a slice knot that is not ribbon.

3.3. Concordance between two knots

Using the notion of sliceness we define a new equivalence relation on the set of knots. Orientations are important. We assume that knots, the ambient space and all submanifolds are oriented and maps are orientation preserving, even though we may not always specify the particular orientation.

DEFINITION 3.3.1. Two knots K_1 and K_2 are said to be concordant if $K_1 \# -K_2$ is slice, where $-K_2$ denotes the mirror image of K_2 with its orientation reversed (see 2.3.2). We write $K_1 \sim K_2$.

THEOREM 3.3.2. Two knots K_1, K_2 are concordant if and only if they cobound a smooth 2-manifold C diffeomorphic to $S^1 \times I$ in $S^3 \times I$ such that $C \cap (S^3 \times \{+1\}) = K_1$ and $C \cap (S^3 \times \{-1\}) = K_2$. Here $I = [-1, 1]$.

Proof: Let $C \subset S^3 \times I$ be a smooth, proper embedding of a cylinder $S^1 \times I$ such that $C \cap (S^3 \times \{+1\}) = K_1$ and $C \cap (S^3 \times \{-1\}) = K_2$. As C has a tubular neighborhood, we have an embedding $\phi\colon (S^1 \times D^2) \times I \to S^3 \times I$ which restricts to $S^1 \times \{0\} \times I$ to give C and intersects the boundary 3-spheres in tubular neighborhoods of the knots. We may assume that after a self-diffeomorphism of $S^3 \times I$, there is an arc $A \subset S^1$ such that $\phi(A \times \{0\} \times \{+1\}) \subset K_1$ and $\phi_{|(A \times D^2) \times I}$ is the product of the inclusion $A \times D^2 \hookrightarrow S^3$ with id_I. Now remove $((A \times D^2) \times I, A \times \{0\} \times I)$ from $(S^3 \times I, C)$ to get the slice disk in B^4 bounded by $K_1 \# - K_2 \in \partial B^4$.

Conversely, [7] suppose that $K_1 \# - K_2$ bounds a disk D in the 4-ball. By adding to this disk a band as in Figure we see and embedded annulus in the 4-ball bounded by the split union of K_1 with $-K_2$. Now add to B^4 a 3-handle along a sphere separating K_1 and $-K_2$. The resulting 4-manifold is $S^3 \times [0, 1]$.

□

THEOREM 3.3.3. The following statements hold:

(1) Knot concordance is an equivalence relation on the set of knots in S^3.
(2) Isotopic knots are concordant.
(3) Connected sum of knots induces a well-defined binary operation on the equivalence classes and under this operation the knot concordance classes form an abelian group, with the class of the unknot representing the identity element, and $-K$ the inverse of the class of knot K.

Proof: With the charaterization of knot concordance as given by Theorem 3.3.2, it is easy to see that this is an equivalence relation on the set of knots in S^3. Reflexivity is seen by taking the concordance to be $K \times I \subset S^3 \times I$, symmetry by turning the concordance between K_1 and K_2 "upside down," and transitivity by "stacking" the concordances between $K_1,\ K_2$ and $K_2,\ K_3$.

If two knots $K_1,\ K_2$ are isotopic, the desired concordance is the image of $K_1 \times I$ under the ambient isotopy F discussed in Section 2.3 following Definition 2.3.1.

As observed in Remark 2.4.1, connected sum defines an associative and commutative binary operation on the isotopy classes of knots and the identity element is the class of the unknot. We must show that this operation is well defined on concordance classes and that the inverse of any K is $-K$. To see that connected sum is well-defined, suppose that $J_1 \sim J_2$ and $K_1 \sim K_2$ [8]. This means that $K_1 \# - K_2$ and $J_1 \# - J_2$ each bound slice disks. By adding a band between these two disks as in Figure , we see that $(K_1 \# - K_2) \# (J_1 \# - J_2)$. As connected sum is associative and commutative,

[7]

[8] This one is easier to do in terms of the definition.

$(K_1\#J_1)\#(-K_2\#-J_2) = (K_1\#J_1)\#(K_2\#J_2)$ is slice, proving the connected sum is well defined on concordance classes. In order to see that $-K$ is the inverse to K we need to show that $K\#-K$ is concordant to the unknot. This follows, since $(K\#-K)\#-U$ is isotopic to $K\#-K$, which is slice by Theorem 3.1.1.

It follows that the set of concordance classes forms a commutative group under the operation of connected sum. $\square$

DEFINITION 3.3.4. The classical knot concordance group $\mathcal{C}$ is the abelian group of concordance classes under the operation of connected sum. It is sometimes denoted by $\mathcal{C}_1$. The identity element in $\mathcal{C}$ is the class of the unknot which consists of slice knots.

As the set of all knots is countable, there being finitely many knots for each positive integer as the minimal crossing number, the group $\mathcal{C}$ is countable.

REMARK 3.3.5. Note that the definition of knot concordance, and in fact all of may be given in the topologically locally flat setting as well. When we want to discuss the topological concordance group we will indicate it by $\mathcal{C}_{top}$ and then we may write the smooth concordance group as $\mathcal{C}_{smooth}$ or just $\mathcal{C}$. The obvious map from $\mathcal{C} \to \mathcal{C}_{top}$ sending the smooth concordance class of a knot K to the topological concordance class of K is a group homomorphism which is onto. As mentioned at the beginning of this chapter, there are several examples which show that this map is not one-to-one. See [**?**, **?**, **?**, **?**, **?**, **?**, **?**].

Gordon introduced the notion of ribbon concordance in [**?**]. A concordance $\mathcal{C} \subset S^3 \times I$ between knots K_1 and K_2 is called a ribbon concordance from K_1 to K_2 if the restriction to $\mathcal{C}$ of the projection $S^3 \times I \to I$ is a Morse function with no local maxima. In this case we write $K_1 \geq K_2$. The relation $\geq$ is clearly reflexive and transitive. Gordon proved that it is in fact a partial ordering on a large set of knots that includes all fibered knots. See [**?**, **?**] for additional work in this area. The following is an observation of Casson.

THEOREM 3.3.6. For each slice knot K, there is a ribbon knot J such that $K\#J$ is ribbon.

Proof: We can arrange the slice disk for K so that the critical points occur in order of index. In addition, the order of appearance of the index 1 critical points can be modified so that after some of the index 1 critical points appear, the cross-section is connected, forming a knot J'. Observe that J' is ribbon and that $K\#-J'$ is also ribbon. This last fact follows from using the obvious concordance from $K\#-J'$ to $K\#-K$ along with the fact that $K\#-K$ is ribbon. □

3.4. Knot genera

Slice genus $g_4(K)$, concordance genus $g_c(K)$

Recall that the standard genus $g(K)$ of a knot K is defined to be the minimal genus of an embedded orientable surface in S^3 with boundary K, and this is well-defined for the isotopy class of K. Sometimes the standard genus is denoted by g_3. The concordance genus $g_c(K)$ is the minimal (standard) genus among all knots in the concordance class of K. The 4-ball genus $g_4(K)$ of a knot K, also known as the slice genus, is defined to be the minimal genus among properly embedded orientable surfaces in the 4-ball with boundary K.

THEOREM 3.4.1. For all knots K, $g_4(K) \leq g_c(K) \leq g(K)$.

Note that the result holds in the topological category as well. If the distinction becomes important, we will keep in mind that the smooth genus is greater than or equal to the topological genus.

Proof of Theorem 3.4.1: First of all, since K is concordant to itself, $g_c(K) \leq g(K)$.

Now, if a knot K has genus g, then there is a Seifert surface of genus g in S^3 bounded by K. The interior of a Seifert surface may be pushed into the interior of the 4-ball to obtain a properly embedded surface of genus g. (See 3.1.11.) This proves that $g_4 \leq g$.

If J and K are concordant and if either of them bounds a genus n properly embedded surface G in B^4, by "stacking" the concordance over G we see a genus n surface bounded by the other knot in B^4. This shows that $g_4(J) \leq g_4(K)$, and $g_4(K) \leq g_4(J)$ rendering them equal to each other.

Now, there exists a knot J concordant to K such that $g_c(K) = g(J)$. So, we have $g_4(K) = g_4(J) \leq g(J) = g_c(K)$, proving that $g_4 \leq g_c$, which completes hte proof. □

Clearly, as a slice knot bounds an embedded disk in the 4-ball and is concordant to the unknot, for a slice knot we have $g_4 = g_c = 0$. Nontrivial slice knots provide examples of knots for which $0 = g_4 = g_c < g$. On the other hand, the trefoil and the Figure 8 knot are examples of non-slice genus one knots. So, for each of these we have $g_4 = g_c = g = 1$.

THEOREM 3.4.2. If a knot K can be unknotted by changing n crossings, then we have $g_4(K) \leq n$. More generally, if K bounds an immersed disk in B^4 with n transverse double points, then $g_4(K) \leq n$.

Proof: Suppose that the knot K has a diagram which transforms into that for an unknot after crossing changes in n places. Each of the crossing changes can be brought on by a sequence of two band moves as shown below.

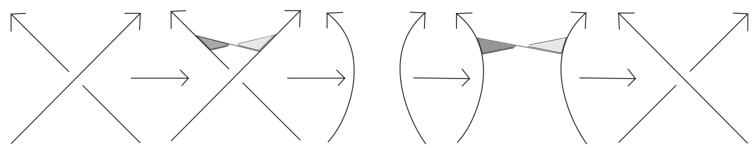

FIGURE 3.12. Crossing change via band moves

Each such sequence corresponds to the following surface in the 4-ball:

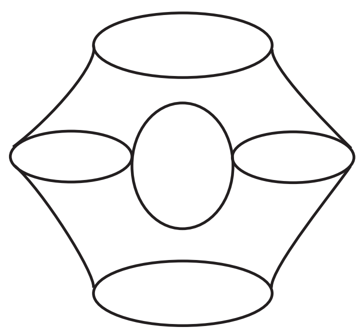

FIGURE 3.13. A surface in B^4 corresponding to a crossing change band move

A sequence of n crossing changes gives a tower of these, and finally, the unknotted circle bounds a disk. This is a genus n surface bounded by the knot, properly embedded in the 4-ball. It follows that $g_4(K) \leq n$. □

3.4.3. **Unknotting Number**

The unknotting number of a knot K is the least number of crossing changes that must be made in any diagram of K to convert it to an unknot. It follows from the above Theorem 3.4.2 that

$$g_4(K) \leq u(K).$$

A related invariant that is more closely tied to concordance was introduced by Askitas [?], which we call the slicing number of a knot: $u_s(K)$ is the minimum number of crossing changes required to convert a knot into a slice knot. It is relatively easy to see that the 4–ball genus of a knot provides a lower bound on the slicing number; it was shown by Livingston in [?] that $u_s(7_4) = 2$, whereas it is known that $g_4(7_4) = 1$.

Livingston also defined a stronger lower bound for the slicing number [?] by taking into account the signs of the crossings when changes are performed. This bound is always at least as good as the slice genus, and the number of negative crossings involved is at least half the signature of the knot. Owens [?] obtained an infinite family of knots for which Livingston's invariant is strictly larger than the slice genus.

We will return to g_4 and g_c in examples in Section 4.4.

3.4.4. **Stable 4-genus** g_{st}

The stable 4-genus of a knot g_{st} is defined as:

$$g_{st}(K) \;=\; \lim_{n\to\infty} \frac{g_4(nK)}{n}.$$

This induces a semi-norm on the rationalized knot concordance group, $\mathcal{C}_{\mathbb{Q}} = \mathcal{C} \otimes \mathbb{Q}$. Basic properties of g_{st} are developed in [?]. This invariant is not always an integer.

3.5. Other notions of concordance

Homology Concordance Group A 3-manifold Σ is called a homology 3-sphere if $H_i(\Sigma, \mathbb{Z}) = H_i(S^3, \mathbb{Z})$ for all values of i. Two homology 3-spheres, Σ_1 and Σ_2, are called homology cobordant if there is a compact 4-manifold W with $\partial W = \Sigma_1 \amalg -\Sigma_2$ and the inclusions of the Σ_i into W induce isomorphisms on homology. The set of homology cobordism classes of homology spheres forms an abelian group under the connected sum operation on 3-manifolds. Work of Freedman implies that in the topological setting the group is trivial; every homology 3-sphere is the boundary of a homology 4-ball. On the other hand, thanks to the work of Donaldson, it is known that in the smooth setting the group is quite large, in that it contains elements of infinite order. (Prior to Donaldson's work it was known only that there is a homomorphism, called the μ *invariant*, of the homology cobordism group onto $\mathbb{Z}_2$. Its defintion depends on Rochlin's theorem.)

Knots K_1 and K_2 in homology spheres Σ_1 and Σ_2 are called homology concordant if they cobound an embedded annulus in a homology cobordism from Σ_1 and Σ_2. The set of homology concordance (or cobordism) classes of knots forms a group under the connected sum operation. This homology concordance group naturally splits as the direct sum over the set of homology cobordism classes of the ambient three-manifolds. If we let $\mathcal{C}^H$ denote the homology concordance group of knots in (the smooth homology cobordism class of) S^3, then there is a well defined homomorphism $\mathcal{C} \to \mathcal{C}^H$. This homomorphism is easily seen to be surjective. It is unknown whether it is injective. The question of injectivity is equivalent to the following:

Open Question 3.5.1. If a knot in S^3 bounds an embedded disk in a homology 4-ball, does the knot bound a slice disk in B^4?

As most of our results apply to the study of $\mathcal{C}^H$, they cannot provide an answer to this question.

Double null concordance A knot is called doubly null concordant if it is the slice of some unknotted 2-sphere in S^4. Note here that every slice knot is the slice of a 2-sphere, possibly knotted, which is obtained by doubling the slice disk. A key result is a consequence of Zeeman's twist spinning construction [?]; every knot of the form $K\#-K$ is doubly null concordant. (Compare the spinning construction we described in the proof of Theorem 3.1.1.)

An easy exercise shows that knots K_1 and K_2 are concordant if and only if $K_1\#J_1 = K_2\#J_2$ for some slice knots J_1 and J_2. Similarly, an equivalence relation can be defined by setting knots K_1 and K_2 doubly concordant if $K_1\#J_1 = K_2\#J_2$ for some doubly null concordant knots J_1 and J_2. There is a resulting group, denoted $\mathcal{C}^D$, induced by connected sum.

A definition of $\mathcal{C}^D$ along the lines of that for $\mathcal{C}$ is as of yet inaccessible. The difficulty is that it is unknown whether the following conjecture is true.

Conjecture 3.5.2. If knots K and $K\#J$ are doubly null concordant, then J is doubly null concordant.

3.5.3. **Concordances to prime knots**

Kirby and Lickorish [?] first proved that every knot is concordant to a prime knot. A simpler proof was developed in [?]. Here is a brief description of such a construction. Figure 3.14 illustrates a knot J contained in a solid torus.

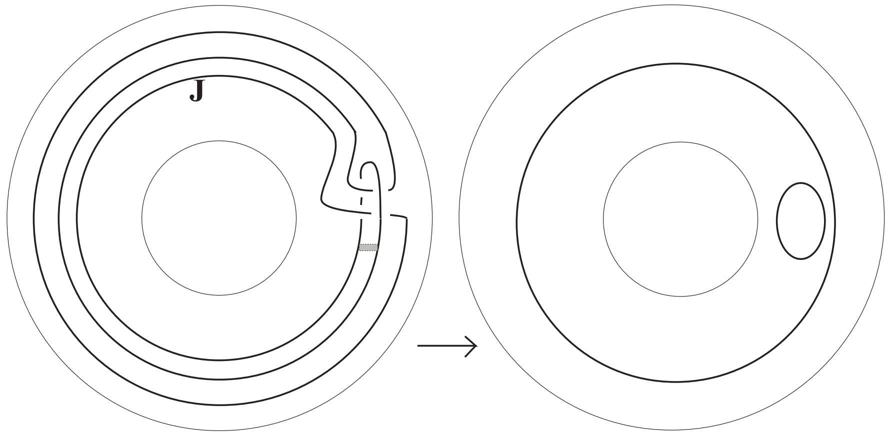

FIGURE 3.14. A knot in a solid torus: the indicated band move splits the knot J into 2 components

For any knot K, the image of J under an embedding carrying the solid torus to a tubular neighborhood of K is a new knot, say K'. The claim is that K' is concordant to K and is prime, as long as K is non-trivial. (If K is trivial, first construct a concordance to a nontrivial knot, for example the Stevedore's knot.) The essential step in constructing the concordance from K' to K is to perform a band move (corresponding to a critical point of index 1 in the concordance) indicated by the gray band in Figure 3.14. This move splits the image of J into an unknot and a copy of K. To prove that K' is prime, consider an embedded 2-sphere meeting the knot in 2 points. An innermost circle argument can be applied to eliminate any intersections of the 2-sphere with the obvious torus in the complement of K'. With these intersections gone, it is easy to argue that the intersection of a ball bounded by the 2-sphere within the solid torus intersects K' in a trivial arc; in short, under the standard embedding of the solid torus in S^3 the image of J is unknotted.

A more sophisticated argument of Myers [**?**] shows that every concordance class contains a hyperbolic knot. The proof is far more difficult in that if a knot is hyperbolic, there are no incompressible tori in its complement. The example constructed above contains an obvious incompressible torus.

CHAPTER 4

Algebraic Concordance

We saw in Section 2.7 that every knot bounds a Seifert surface in S^3. Using a similar argument as in the proof of Theorem 2.7.2, we will now show that if a knot is slice, its Seifert surface and slice disk together bound an orientable 3-dimensional manifold inside the 4-ball. As a consequence, there is an obstruction to sliceness arising from the "half lives, half dies" principle of Theorem 2.6.1, applied to the Seifert form of a slice knot (Theorem 4.2.1). In the rest of this chapter, using this obstruction we will obtain necessary conditions for sliceness in terms of the algebraic invariants defined in Section 2.10.

We saw in Section 2.7 that every knot bounds a Seifert surface in S^3. The resulting s-equivalence class of Seifert matrices provides an avenue to several invariants of knots: the Alexander polynomial, the Arf invariant, the Levine tristram signature, and so on. All of these provide invariants of knot concordance. At its heart Algebraic concordance determines what information about knot concordance is carried by the Seifert matrix.

This Chapter and the two after concentrate on Algebraic concordance. The first two sections culminate Theorem 4.2.1 which says that a $2g \times 2g$ Seifert matrix for a slice knot has the form $\begin{pmatrix} 0^{g\times g} & A \\ B & C \end{pmatrix}$ (up to a change of basis). Such knots are called algebraically slice. In sections 4.3 and 4.4 we connect this obstruction to the Alexander polynomial, signature and Arf invariant, and then to the study of knot genera. A formal definition of algebraic concordance appears in Section 4.5. Unlike the classical concordance group (either smooth or topological) the algebraic concordance group can be computed. This is done in the celebrated work of J. Levine [**?**, **?**] by exhibiting an injection to a larger group, for which a complete set of invariants can be constructed. We set this up at the end of this Chapter. In Chapter 5 we extract from this group invariants by recalling the theory of Witt groups of bilinear forms and close with a summary of Levine's proof. In Chapter 6 we study phenomena involving order in the algebraic concordance group.

4.1. A Seifert surface and a slice disk cobounding a 3-manifold

Recall the review of obstruction theory and transversality from 2.7.1. The manifold X of interest to us in this section is the complement or the (compact) exterior of a 2-dimensional slice disk properly and smoothly embedded in the 4-ball.

THEOREM 4.1.1. If $(S^3, K) = \partial(B^4, D)$ is a slice knot bounding a Seifert surface $F \subset S^3$ and a slice disk $D \subset B^4$, then there is a compact orientable 3-manifold M in B^4 with $\partial M = D \cup_K F$.

Proof

Let K, D and F be as in the statement of the theorem. Let $N(D)$ be a tubular neighborhood of the slice disk D, such that the homeomorphism $\phi\colon D^2 \times B^2 \to N(D)$ extends $\phi\colon S^1 \times B^2 \to N(K)$, where $N(K)$ is a tubular neighborhood of K.

We may further assume that $\phi(D^2 \times 0) = D$, and as in Section 2.7, $F \cap N(K) = \phi(\partial D^2 \times [0,1])$; $l_K = F \cap \partial N(K)$ corresponds to $\partial D^2 \times 1$, and the meridian m_K is $\phi(1 \times \partial B^2)$.

Let $f\colon S^3 - K \to S^1$ be as in the proof of Theorem 2.7.2. That is, $f^{-1}\{1\} = \mathrm{Int}(F)$ and $f_{|N(K)-K}(\phi(z,w)) = \frac{w}{|w|}$.

Our goal is to extend this f to a map $g\colon B^4 - D \to S^1$, such that $g^{-1}(1) \cup K$ is the desired orientable, 3-manifold M. To this end, first define g on $N(D) - D$ as $g\,(\,\phi(z,w)\,) = \frac{w}{|w|} \in S^1$, and $g_{|(S^3-K)} = f$.

Recall that the slice disk exterior is $X(D) = B^4 - \mathrm{Int}(N(D))$, and $\partial X(D) = M(K,0)$, the 0-surgery on S^3 along the knot K. (See Lemma 3.2.1.) Specifically,

$$\partial X(D) \;=\; X(K) \bigcup_{\partial X(K) \cong \phi(\partial D^2 \times \partial B^2)} \phi(D^2 \times \partial B^2),$$

where the null-homologous longitude $l_K \subset X(K)$ is identified with the homologically trivial meridian $\phi\,(\partial D^2 \times 1)$ of the solid torus $\phi(D^2 \times \partial B^2)$ attached to $X(K)$.

To extend $g_{|\partial X(D)}$ to all of $X(D)$, next, we define g arbitrarily on the relative 0-skeleton of $(X(D), \partial X(D))$, and since S^1 is path-connected, it extends to the 1-skeleton. Obstruction for an extension to the 2-skeleton lies within $H^2(X(D), \partial X(D); \mathbb{Z})$. By Poincaré-Lefschetz duality, this is isomorphic to $H_2(X(D))$ which vanishes as seen in Remark 3.2.3.

For $k \geq 3$, as $\pi_{k-1}(S^1) = 0$, we have $H^k(X(D), \partial X(D); \pi_{k-1}(S^1)) = 0$, so again the obstruction vanishes.

We now have a map $g\colon (B^4 - D) \to S^1$ and via transversality, after a small perturbation if necessary, g is a smooth map with 1 as a regular value. It follows that $g^{-1}(1)$ is an oriented codimension one submanifold of $B^4 - D$. If $g^{-1}(1)$ is disconnected, we discard the extraneous closed components. Closure of the remaining component in B^4 is a compact orientable 3-manifold bounded by $D \cup_K F$. □

4.2. Seifert matrix of a slice knot

THEOREM 4.2.1. If K is a slice knot, F a Seifert surface for K, and θ the Seifert form, then there exists a summand (metabolizer) H of $H_1(F)$ such that

(1) $\mathrm{rk}(H) = \frac{1}{2}\mathrm{rk}(H_1(F))$,
(2) $\theta_{H\times H} = 0$.

Proof From Theorem 4.1.1 we have a compact orientable 3-manifold M in B^4 with $\partial M = D \cup_K F$. Consider the map $H_1(F;\mathbb{Q}) \cong H_1(F \cup D;\mathbb{Q}) \to H_1(M;\mathbb{Q})$ induced by inclusion. By Theorem 2.6.1, the kernel N is a half dimensional subspace of $H_1(F;\mathbb{Q})$.

Given any two classes $x, y \in N$, multiples ax and by are in the kernel of the inclusion $H_1(F) \to H_1(M)$. Representing x and y by curves α and β, we see that $a\alpha$ bounds a surface in M, and the positive push-off $i_+(b\beta)$ (see Definition 2.8.1) bounds a surface in $B^4 - M$. These are disjoint, so $\mathrm{lk}(ax, i_+(by)) = 0$. Since the linking form takes values in $\mathbb{Z}$ and is bilinear, this implies that $\mathrm{lk}(x, i_+(y)) = 0$. That is, the Seifert form vanishes on the pair (x, y), as desired. □

Conditions (1) and (2) described in Theorem 4.2.1 naturally lead to the following notion of triviality for a Seifert matrix.

DEFINITION 4.2.2. A $2n \times 2n$-dimensional Seifert form is algebraically slice or metabolic if there is an n-dimensional summand of the underlying free $\mathbb{Z}$–module on which the form vanishes. Such a summand is called a metabolizer of the form.

Equivalently, a $(2n \times 2n)$ dimensional Seifert matrix A is algebraically slice if it is congruent to a matrix with a half dimensional block of zeros. That is, there exists a nonsingular integral matrix P such that

$$PAP^T = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$$

where the top left 0 is an $n \times n$ diagonal block of 0s.

Definition 4.2.3. A knot K is called algebraically slice if it has a Seifert matrix which is algebraically slice.

Recall that any two Seifert matrices for a knot are S-equivalent (see 2.9.5). It is true that if some Seifert matrix for K is algebraically slice, then all are, independent of the choice of Seifert surface. In order to prove this, we will need results of Section 4.5. See Remark 4.5.6.

The following corollary is a direct consequence of Theorem 4.2.1. As will be seen later in the book, its converse is not true.

Corollary 4.2.4. A slice knot is algebraically slice.

Exercise 4.2.5. In Exercise 2.8.6 you were asked to find a Seifert matrix for Stevedore's knot 6_1. In Figure 3.8 we saw that this knot is ribbon and therefore slice. Find a metabolizer of the Seifert form of 6_1. Represent the metabolizer in terms of the curves on the surface.

Exercise 4.2.6. Suppose that K bounds a Seifert surface $F \subset S^3$ of genus g and bounds a surface $G \subset B^4$ of genus g', $g' \leq g$. An argument as in the proof of Theorem 4.2.1 shows that $F \cup G$ bounds a 3–manifold $M \subset B^4$. Prove that in this situation, the $2g \times 2g$ Seifert form V with respect to some basis for $H_1(F)$, is of the form

$$\begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$$

where 0 is the $(g - g') \times (g - g')$ matrix of 0s.

Exercise 4.2.7. Show that if a knot is doubly null concordant (see 3.5 for definition) then for some Seifert surface F, the associated Seifert form has metabolizers M and M' with $M \oplus M' = H_1(F, \mathbb{Z})$. Such a form is called *hyperbolic*.

Exercise 4.2.8. Show that for a genus one Seifert surface, the two dimensional Seifert form A is algebraically slice (metabolic) if and only if $1 - 4 \cdot \det(A)$ is a perfect square. Also show that in such a case A has two, rank 1 metabolizers.

Exercise 4.2.9. Consider the Seifert matrix $A = \begin{pmatrix} 0 & a+1 \\ a & b \end{pmatrix}$. In this case, we have $1 - 4 \cdot \det(A) = (2a + 1)^2$, and therefore by Exercise 4.2.8, A has two, rank one metabolizers H and H'. Find a generating vector for H and another one for H'. Now, show that if $(2a + 1) \nmid b$, then these metabolizers do not satisfy $H \oplus H' = H_1(F)$.

Note that we cannot a priori conclude from Exercise 4.2.9 that a knot with this Seifert matrix is not doubly null concordant. From Exercise 4.2.7, we only know that for *some* Seifert surface for such a knot, the Seifert form will split along metabolizers. The condition that a doubly null concordant knot has a hyperbolic Seifert matrix is not as strong as the condition for algebraically slice (null concordant) knots. In case of an algebraically slice knot *every* Seifert matrix is algebraically slice. It is however true that in case of this particular matrix, no knot with this Seifert matrix is doubly null concordant. Proving this requires more sophisticated tools. We will revisit this Seifert form in 7.5.2 and 7.10.3.

4.3. Algebraically slice knots

In Section 2.10 we defined algebraic invariants of knots, namely, the determinant, Alexander polynomial, Arf invariant and signatures. Since all of these are defined in terms of Seifert matrices, algebraic sliceness forces them to have special properties. We describe these below through a series of exercises, all of which could be solved using the special form that matrices of algebraically slice knots have, as described in Definition 4.2.2 and Corollary 4.2.4.

Determinant

Exercise 4.3.1. Prove that if K is algebraically slice and A is a Seifert matrix of K, then the absolute value of the determinant of K, $|\det(A + A^T)|$, is a square of a nonzero integer.

Alexander Polynomial

Recall that the Alexander polynomial for a knot K with Seifert matrix A is $\Delta_K(t) = \det(A - tA^T)$, defined in 2.10.2.

Exercise 4.3.2. Prove that if K is algebraically slice, then the Alexander polynomial of K is of the form $\Delta_K(t) = \pm t^k f(t) f(t^{-1})$ for some polynomial $f(t) \in \mathbb{Z}[t]$ with $f(1) = \pm 1$.

As $\det(K) = \Delta(-1)$, the statement in Exercise 4.3.1 also follows from Exercise 4.3.2.

Example 4.3.3. The Alexander polynomials of the trefoil and the figure-8 knot are $t^2 - t + 1$ and $t^2 - 3t + 1$, respectively. Neither of these factors as above. Therefore the trefoil and the figure 8 knot cannot be slice knots. Moreover, from Exercise 2.3.5 (also see 3.1.2) it now follows that the Figure 8 knot represents an order two class in the knot concordance group $\mathcal{C}$ described in 3.3.4.

Exercise 4.3.4. Verify directly by factoring that the Alexander polynomial of Stevedore's knot satisfies the algebraically slice polynomial condition in Corollary 4.3.2.

Exercise 4.3.5. Identify knots with ten crossings or less which do not satisfy the slice polynomial criterion in 4.3.2 and therefore cannot be slice.

Exercise 4.3.6. Show that a quadratic Alexander polynomial for an algebraically slice knot has the form $nt^2 - (2n+1)t + n$, where $n = k(k-1),\ k \in \mathbb{N}$.

Exercise 4.3.7. Suppose that $\delta(t)$ is an irreducible symmetric integral polynomial, i.e., $\delta(t) = \pm t^k \delta(t^{-1}))$, and $\delta(t)$ has odd exponent in $\Delta_K(t)$. That is, δ^m divides Δ_K for some odd m but δ^{m+1} does not divide Δ_K. Show that K is not slice.

Arf Invariant

The $\mathbb{Z}/2$-valued Arf invariant was defined in 2.10.8. Murasugi [**?**, Theorem 2] observed that $\mathrm{Arf}(A) = 0$ if and only if the Alexander polynomial Δ evaluated at -1 is $\pm 1 \mod 8$. We know that $\Delta(-1)$ is odd, and if the Seifert form is algebraically slice, it is also a square. (See Exercise 4.3.1.) Since squared odd integers are $1 \mod 8$, it follows that the Arf invariant vanishes on metabolic forms.

Hence, the Arf invariant gives a well defined $\mathbb{Z}/2\mathbb{Z}$-valued homomorphism from $\mathcal{C}$ to $\mathbb{Z}/2\mathbb{Z}$.

Signatures

Recall the definition of knot signatures from 2.10.9.

Corollary 4.3.8. If the knot K is algebraically slice, ω is a unit complex number and $\Delta_K(\omega) \neq 0$, then the ω signature $\sigma^*_\omega(K)$, defined in 2.10.13, is 0.

Proof Suppose that A is a $2n \times 2n$ algebraically slice Seifert matrix for K, $\omega \neq 1$, and recall that $\sigma^*_\omega(K)$ is defined as the signature of the hermitian matrix $A_\omega = (1-\omega)A + (1-\bar{\omega})A^T$. Let $a = (1-\omega)^{2n}$. Then $a \neq 0$ and we have:

$$\det(A_\omega) = a \det\left(A + \frac{(1-\bar{\omega})}{(1-\omega)} A^T\right) = a \det(A - \bar{\omega} A^T) = a \Delta_K(\bar{\omega}).$$

It follows that A_ω is nonsingular if $\bar{\omega}$ (and consequently ω) is not a root of the Alexander polynomial. Therefore under the given hypothesis, by performing simultaneous row and column operations on A_ω we obtain a congruent, hermitian matrix of the form $\begin{pmatrix} 0 & B \\ \bar{B}^T & 0 \end{pmatrix}$, where B is nonsingular. Additional matrix operations establish congruence with $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, which in turn is congruent to the direct sum of n many copies of the the 2×2 form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Finally, we diagonalize: Let $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. We have

$$P \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P^T = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix},$$

which has signature 0. □

Exercise 4.3.9. Show that if K is algebraically slice, then the Tristram-Levine signature $\sigma_\omega(K)$, defined in 2.10.16 as $\lim(\sigma^*_{\omega^+}(K) + \sigma^*_{\omega^-}(K))/2$, is 0 for each ω with unit norm.

As these signatures are additive for connect sums of knots, and are negatives of each other for concordance inverses, it follows that $\sigma_\omega \colon \mathcal{C} \to \mathbb{Z}$ is a group homomorphism.

Exercise 4.3.10. Show that if a knot K has finite concordance order, i.e., if there is a positive integer n such that nK is slice, then $\sigma_\omega(K) = 0$ for all unit $\omega \in \mathbb{C}$.

Exercise 4.3.11. Show that the trefoil knot has infinite concordance order.

The knot shown below is called a twist knot. The box labeled $2m$ indicates that there are $2m$ half- or m full-twists between the two strands. A negative twist in this case looks like the one on the right of the box. Note that K_1 is the left handed trefoil and K_{-1} is the Figure 8 knot shown in 2.3.5.

Exercise 4.3.12. Answer the following questions for the knot K_m in Figure 4.1.

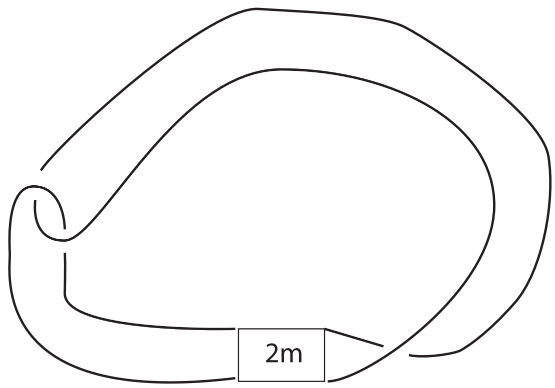

FIGURE 4.1. The twist knot K_m

(1) The knot bounds an obvious genus one Seifert surface F. Find an appropriate basis for $H_1(F)$ such that the Seifert form is represented by the matrix:

$$A_m = \begin{pmatrix} 1 & 1 \\ 0 & m \end{pmatrix}.$$

(2) using Exercise 4.3.6 identify values of $m \in \mathbb{Z}$, for which the Alexander polynomial $\Delta_m(t)$ satisfies the slice condition given in 4.3.2. For each such value of m, find a metabolizing vector for the Seifert form. That is, find $(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}$ such that

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} A_m \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0.$$

(3) Stevedore's knot 6_1 (Figure 3.1) belongs to this family. What is the value of m for 6_1?

(4) Compute the signature $\sigma(K_m)$. (See 2.10.10 for definition.)

(5) Let S be the set of values of m for which $\sigma(K_m) \neq 0$. Show that no combination $\#_m K_m$ is slice, if the connected sum is taken over a finite subset of S.

EXERCISE 4.3.13. Construct an infinite linearly independent set of knots in $\mathcal{C}$.

EXERCISE 4.3.14. Show that the Tristram-Levine signatures σ_ω give a map of $\mathcal{C}$ onto $\oplus_1^\infty \mathbb{Z}$.

EXERCISE 4.3.15. For $K = 4_1$, the Figure 8 knot, explicitly compute the signature function $\sigma_K \colon S^1 \to \mathbb{Z}$ given by $\sigma_K(\omega) = \sigma_\omega(K)$ using Seifert matrices.

In Example 4.3.3 we saw that the Figure 8 knot has order 2 in $\mathcal{C}$. A disk with bands illustration for this knot, as in the right side of Figure 2.25, has $+2$ full twists in the left band, -2 full twists in the right band, and in the middle the left band crossing once over the right band. This gives the Seifert matrix:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

The following Exercise generalizes this example.

Exercise 4.3.16. Find negative amphicheiral knots with Seifert matrices of the form $\begin{pmatrix} a & 1 \\ 0 & -a \end{pmatrix}$. Use these to construct an infinite family of elements of order 2 in $\mathcal{C}$.

Exercise 4.3.17. Use Exercises 4.3.13 and 4.3.16 to show that $\mathcal{C}$ contains a *summand* isomorphic to $(\oplus_1^\infty \mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/2\mathbb{Z})$. Equivalently, show that there is a split exact sequence:

$$0 \to \mathrm{Ker}(\psi) \to \mathcal{C} \xrightarrow{\psi} (\oplus_1^\infty \mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/2\mathbb{Z}) \to 0.$$

4.4. Revisiting knot genera

In 3.4 we discussed various notions of knot genera and in Theorem 3.4.1 we showed that $g_4(K) \leq g_c(K) \leq g(K)$. The following examples from [?] make effective use of the Alexander polynomial and signature conditions To exhibit examples for which these inequalities are strict.

Example 4.4.1. If $K = 6_2$, [1] we have: $g(K) = 2,\ g_c(K) = 2,\ g_4(K) = 1$.

A Seifert surface for 6_2 of genus 2 is easily found, which gives $g(6_2) \leq 2$. Since $\Delta_{6_2}(t) = 1 - 3t + 3t^2 - 3t^3 + t^4$, we have $g(6_2) \geq 2 = \frac{1}{2}\deg(\Delta)$. So $g(6_2) = 2$.

Next, since K has unknotting number one, by 3.4.3, we have $g_4(6_2) \leq 1$. Since Δ_{6_2} is irreducible, by 4.3.2 it follows that 6_2 is not slice, so $g_4(6_2) \neq 0 \Longrightarrow g_4(6_2) = 1$.

Finally, one sees that $g_c(6_2) = 2$ as follows. If 6_2 is concordant to J with $g(J) \leq 1$, then $\deg(\Delta_J) \leq 2$. Moreover, $6_2\# - J$ is slice and by 4.3.2, $\Delta_{6_2}(t)\Delta_J(t) = t^d f(t) f(t^{-1})$ for some $f(t)$. But since $\Delta_{6_2}(t)$ is irreducible and of degree 4, while $\deg(\Delta_J(t)) \leq 2$, this is not possible.

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EXAMPLE 4.4.2. Casson has made the following interesting observation: the knot $K = {7_6}^2$ has $g_4(K) = 1$ but it is not concordant to a genus 1 knot. To prove this, one needs only the following facts.

The knot 7_6 has unknotting number 1 and has irreducible Alexander polynomial $t^4 - 5t^3 + 7t^2 - 5t + 1$. The inverse $-K = rmK$ of a genus one knot K would have genus one. The Alexander polynomial of a genus one knot will have degree two or zero. Moreover the Alexander polynomial multiplies under connected sum and is unchanged by changes in the orientation of the knot or S^3. It follows that we cannot form a connected sum of 7_6 with the inverse of a genus one knot and get the Alexander polynomial of the connected sum to factor as in 4.3.2.

One can similarly find unknotting number 1 knots that are not concordant to knots of genus n for arbitrary n. Use the fact that any Alexander polynomial occurs as that of some unknotting number 1 knot [**?**].

EXAMPLE 4.4.3. Note that the square knot shown in Figure 3.2 and Figure 3.4 is slice and it has unknotting number 2. So the connected sum of n square knots is slice; also it has unknotting number $2n$. (The homology of the 2-fold cover is $(\mathbb{Z}/3\mathbb{Z})^{2n}$, and the rank of the $\mathbb{Z}/p\mathbb{Z}$ homology is a lower bound on the unknotting number.) Thus, we have a knot K with $g_4 = g_c = 0$, but with $u(K)$ arbitrarily large.

EXAMPLE 4.4.4. If $K = 6_2\#6_2$: $g(K) = 4,\ g_c(K) = 4,\ g_4(K) = 2$.

The knot $6_2\#6_2$ cannot be handled in the same way, since its Alexander polynomial is $\Delta_{6_2\#6_2}(t) = (1 - 3t + 3t^2 - 3t^3 + t^4)^2$, which is, in fact, the Alexander polynomial of the slice knot $6_2\#{-6_2}$. By the additivity of the 3–genus, we do have that $g(6_2\#6_2) = 4$. Introducing the signature function permits the further analysis of this example.

For any knot K, the Murasugi [**?**] 4–genus bound is given by $2g_4(K) \geq |\sigma(K)|$. (See Theorem 7.11.3.) From Example 4.4.1 we have $g_4(6_2) = 1 \implies g_4(6_2\#6_2) \leq 2$; also, $\sigma(6_2\#6_2) = 4$, and so $g_4(6_2\#6_2) = 2$.

The Levine-Tristram signature function of a knot, $\sigma_K(\omega)$, is the function defined on the unit complex circle as the local average of the signature of the hermetianized Seifert form $(1 - \omega A) + (1 - \overline{\omega})A^T$, $\omega \in S^1$. The Murasugi bound (Theorem 7.11.3) generalizes to $2g_4(K) \geq |\sigma_K(\omega)|$, and as a consequence, σ_K is a concordance invariant. (See 4.3.9.)

As seen in Exercise 2.10.14, for a knot K, its signature function $\sigma_K(\omega), \omega \in S^1$, is an integer-valued function. The only possible discontinuities of $\sigma_K(\omega)$ occur at roots of

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$\Delta_K(t)$. For ω near 1, $\sigma_K(\omega) = 0$. Thus, since $\sigma(6_2\#6_2) = \sigma_{6_2\#6_2}(-1) = 4$, we see that $\Delta_{6_2\#6_2}(t)$ must have a root on the unit circle and the signature function has a jump at one such root. (In fact, this polynomial has a unique conjugate pair of unit roots.)

If $6_2\#6_2$ is concordant to J, then the signature function $\sigma_J(\omega)$ must similarly have a jump at a root of $\Delta_{6_2\#6_2}$, and it immediately follows that $1 - 3t + 3t^2 - 3t^3 + t^4$ divides $\Delta_J(t)$. It then follows from the Fox-Milnor theorem 4.3.2 that $(1 - 3t + 3t^2 - 3t^3 + t^4)^2$ divides $\Delta_J(t)$, and so we see that $g_c(6_2\#6_2) = 4$.

4.5. The (integral) algebraic concordance group

As seen in Section 3.3, the concept of sliceness, together with the connected sum operation, allows us to define the equivalence relation of knot concordance. In Theorem 3.3.3 we showed that concordance classes form an abelian group under the operation induced by connected sum of knots. We now use the algebraic sliceness of a Seifert matrix along with block direct sums, or orthogonal sums, of matrices to define a corresponding equivalence relation on Seifert matrices, which we will call algebraic concordance. First, we will generalize the notion of a Seifert matrix to allow us to work over the field of rationals and its field extensions.

Recall from Section 2.8, that an even-dimensional matrix A with entries in $\mathbb{Z}$ is called a Seifert matrix if $\det(A - A^T)$ is a unit in $\mathbb{Z}$, or in other words, $\det(A - A^T) = \pm 1$. In Exercise 2.8.11, we saw that given any such matrix, there is a knot with that as its Seifert matrix. For matrices over a field $\mathbb{F}$ with characteristic not equal to 2, we have a similar definition below.

Definition 4.5.1. An even-dimensional matrix A with entries in a field $\mathbb{F}$ is called a Seifert matrix or an $\mathbb{F}$-Seifert matrix if it satisfies $\det\big((A - A^T)(A + A^T)\big) \neq 0$. The corresponding Seifert form is a bilinear form on a vector space V of dimension equal to the dimension of A, given by $\theta(x, y) = x^T A y$, for $x, y \in V$ written as column vectors.

In [**?**, **?**] such matrices are referred to as *admissible*. Next we define algebraic sliceness for these Seifert forms and matrices, similar to 4.2.2.

Definition 4.5.2. A $2n \times 2n$ dimensional $\mathbb{F}$-Seifert matrix is said to be algebraically slice or metabolic if there is an n-dimensional subspace of $V \cong \mathbb{F}^{2n}$ on which the corresponding form vanishes. That is, there exists an $\mathbb{F}$ matrix P such that $\det(P) \neq 0$

and

$$PAP^T = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$$

where the top left 0 is an $n \times n$ diagonal block of 0s.

The following discussion applies to both integral Seifert matrices as well as those with entries in a field $\mathbb{F}$ with characteristic not equal to 2. We will make a distinction between the two cases only when needed.

DEFINITION 4.5.3. Seifert matrices A_1 and A_2 are called Witt equivalent or, more commonly, algebraically concordant if the orthogonal sum $A_1 \oplus -A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & -A_2 \end{pmatrix}$ is algebraically slice. We denote this by $A_1 \sim A_2$.

THEOREM 4.5.4. Algebraic concordance is an equivalence relation on the set of Seifert matrices.

Proof Let A, B and C be Seifert matrices with $\dim(A) = 2m$, and $\dim(B) = 2n$.

Reflexivity $A \sim A$: Let P be the $4m \times 4m$ dimensional matrix which adds the $(2m+i)$th row of a $4m \times 4m$ matrix to its ith row, for $1 \leq i \leq 2m$.
It is clear that P is nonsingular and $P(A \oplus -A)P^T$ contains a $(2m)$ dimensional diagonal block of 0s.

Symmetry $A \sim B \Rightarrow B \sim A$: Suppose $A \sim B$. By definition, there is a nonsingular matrix P such that $P(A \oplus -B)P^T$ contains an $(m+n)$ dimensional diagonal block of 0s. Let $R_1, R_2, \cdots$ denote the rows of P in order. Let Q be the matrix whose rows are $R_{2m+1}, R_{2m+2}, \cdots, R_{2m+2n}, R_1, R_2, \cdots, R_{2m}$ in this order.
It is easy to check that $Q(B \oplus -A)Q^T$ contains an $(m+n)$ dimensional diagonal block of 0s.

Before proving transitivity, let's make a note that orthogonal sum of algebraic concordance classes of Seifert matrices is associative and commutative. Proof of this is similar to above proofs. Furthermore, with a change of basis that results in appropriate row and column exchanges, it is easy to see that the orthogonal sum of algebraically slice matrices is algebraically slice.

Transitivity ($A \sim B$ and $B \sim C$) $\Rightarrow A \sim C$: Suppose that $A \sim B$ and $B \sim C$. We have:

$$A \oplus (-B \oplus B) \oplus -C \;\sim\; (A \oplus (-C)) \oplus (-B \oplus B)$$

is algebraically slice. By reflexivity, we know that $-B \oplus B$ is algebraically slice. By Lemma 4.5.5 below, called Witt Cancellation, this implies that so is $A \oplus -C$. This completes the proof of transitivity. □

The *Witt cancellation* lemma below was proved in [**?**, Lemma 1] using a dimension counting argument which we will detail below.

Lemma 4.5.5. Let N and A be matrices of dimensions $2k$ and $2m$ respectively. Suppose that N and $A \oplus N$ are algebraically slice and that $N - N^T$ has non-zero determinant. Then A is algebraically slice.

Proof

Let $B = A \oplus N$ be a bilinear form on the vector space $V_{2m} \oplus V_{2k}$, such that $B_{|V_{2m}} = A$, $B_{|V_{2k}} = N$, and B acts orthogonally on the direct sum, meaning for $v \in V_{2m} \oplus 0$ and $w \in 0 \oplus V_{2k}$, we have $B(v, w) = 0$.
Since N is algebraically slice, the second summand V_{2k} splits into $H \oplus V_k$, where $H \cong V_k$ is a metabolizer of N.
Since B itself is algebraically slice, it has a metabolizer. Let $1 \leq i \leq n = m + k$, and let $\alpha_i = (x_i, y_i, z_i)$, where $x_i \in V_{2m},\ y_i \in H,\ z_i \in V_k$, form a basis for this metabolizer.
We have $B(\alpha_i, \alpha_j) = 0$ and $N(y_i, y_j) = 0$, for all i, j.
Consider $z_1, \cdots, z_n \in V_k$.
With a change of basis for V_k we may arrange that $z_1, \cdots, z_r$ are linearly independent and $z_{r+1} = \cdots = z_n = 0$.
Clearly, $r \leq k$ and and for $r + 1 \leq i \leq n$, we have $\alpha_i = (x_i, y_i, 0)$.
Now, similarly working with $x_{r+1}, \cdots, x_n$, we may assume that $x_{r+1}, \cdots, x_{r+s}$ are linearly independent and $x_{r+s+1} = \cdots = x_n = 0$.
It follows that for $r + 1 \leq i, j \leq r + s$, we have:

$$0 \;=\; B(\alpha_i, \alpha_j) \;=\; A(x_i, x_j) \oplus N(y_i, y_j) \;=\; A(x_i, x_j).$$

Thus, $x_{r+1}, \cdots, x_{r+s}$ are metabolizing vectors for A. Our next goal is to show that $s \geq m$. It will then follow that A is algebraically slice.
Note that for $r + s + 1 \leq i \leq n,\ x_i = z_i = 0$. As $(x_i, y_i, z_i) = (0, y_i, 0)$ are linearly independent, clearly $\{\, y_i,\ r + s + 1 \leq i \leq n \,\}$ is a linearly independent set. let these

vectors y_i span a subspace Y of the metabolizer $H \cong V_k$ of N. This space Y has rank $n-r-s$.
Furthermore, for $i \geq r+s+1$, we have

$$0 = B(\alpha_i, \alpha_j) = N\left((y_i, 0), (y_j, z_j) \right) = N\left((y_i, 0), (0, z_j) \right),$$

and similarly, $N\left((0, z_j), (y_i, 0) \right) = 0$. Therefore, Y is orthogonal to the rank r subspace $\langle\, z_1, \cdots z_r \,\rangle$ of V_k under the bilinear form given by the non-singular matrix $N - N^T$.
Extend the basis of Y to a basis of H, and the linearly independent set $\{\, z_i \,|\, 1 \leq i \leq r \,\}$ to a basis for the other V_k summand.
With respect to the resulting basis of V_{2k}, the matrix of $N - N^T$ has a $k \times k$ diagonal block of zeros which corresponds to the basis vectors of the metabolizer H, followed by an $(n-r-s) \times r$ matrix of zeros in the $(n-r-s)$ rows, corresponding to $y_{r+s+1}, \cdots, y_n$, and r columns corresponding to $z_1, \ \cdots, z_r$.
Non-singularity of $N - N^T$ requires that rows of this matrix are independent; in particular, the remaining $k-r$ columns will need to contribute a sufficient number of nonzero entries to make the $n-r-s$ rows, corresponding to $y_{r+s+1}, \cdots, y_n$, linearly independent.
Therefore, $k - r \geq n - r - s$. Since $n = m + k$, we have $s \geq m$. □

Remark 4.5.6. Recall the definition of S-equiavlence of Seifert matrices from Section 2.9. Clearly, a stabilization that adds a row and a column will not change the algebraic concordance class of a Seifert matrix. As by definition, S-equivalent Seifert matrices are related by a sequence of stabilizations and congruences, they are algebraically concordant.

Definition 4.5.7. Equivalence classes of Seifert matrices, with entries in $\mathbb{Z}$ or in a field $\mathbb{F}$, with characteristic not equal to 2, under the relation of algebraic concordance described in 4.5.3 form an abelian group under orthogonal sum. Associativity and commutativity are easy to check. The class of the 0-dimensional "empty" matrix, or equivalently, any $2n \times 2n$ matrix with all 0 entries serves as the identity element. Inverse of the class of A is represented by $-A$. We denote this group by $\mathcal{G}^{\mathbb{Z}}$ or $\mathcal{G}^{\mathbb{F}}$, as appropriate. Typically, $\mathcal{G}^{\mathbb{Z}}$ is written simply as $\mathcal{G}$ and called the (integral) algebraic concordance group; the group $\mathcal{G}^{\mathbb{Q}}$ is called the rational algebraic concordance group. Knots with algebraically concordant (integral) Seifert forms are said to be algebraically concordant.

Note that if A is a Seifert matrix for a knot K, the Seifert matrix for the reverse knot rK is A^T.[3] (Exercise 2.8.7.) As we will revisit in Section 4.6.2, algebraic concordance

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does not distinguish any knot from its reverse. In the meantime, we invite the reader to verify this for some examples.

Exercise 4.5.8. *Recall that the twist knot of Figure 4.1 has Seifert matrix* $A_m = \begin{pmatrix} 1 & 1 \\ 0 & m \end{pmatrix}$. *Verify that* A_m *is algebraically concordant to* A_m^T.

Theorem 4.5.9. The map that sends the concordance class of a knot to the algebraic concordance class of its Seifert matrix is a well-defined group homomorphism from $\mathcal{C}$ onto $\mathcal{G}^{\mathbb{Z}}$.

Exercise 4.5.10. Prove Theorem 4.5.9. To prove surjectivity, recall from Exercise 2.8.11 that given an abstract Seifert matrix, a corresponding knot can be obtained.

Remark 4.5.11. It is worth noting that Theorem 4.5.9 also holds for the topological concordance group $\mathcal{C}_{top}$ mentioned in Remark 3.3.5; proof is identical, and we have

$$\mathcal{C} \overset{\text{onto}}{\to} \mathcal{C}_{top} \overset{\text{onto}}{\to} \mathcal{G}^{\mathbb{Z}}.$$

4.6. Algebraic concordance and properties of knots

4.6.1. Knot Symmetry: Amphicheirality. Recall knot reversibility and amphicheirality properties from Definition 2.3.2. Each of these constrains the algebraic invariants of a knot, and hence can constrain the concordance class of a knot. For instance, according to Hartley [**?**], if a knot K is negative amphicheiral, then its Alexander polynomial satisfies $\Delta_K(t^2) = F(t)F(t^{-1})$ for some symmetric polynomial F. It follows quickly from the condition that slice knots have polynomials that factor as $g(t)g(t^{-1})$ (Exercise 4.3.2) that if a knot K is concordant to a negative amphicheiral knot, $\Delta_K(t^2)$ must factor as $F(t)F(t^{-1})$. Further discussion of amphicheirality and knot concordance is included in [**?**], where the focus is on higher dimensions, but some results apply in dimension three.

Example 4.6.1. Recall the knots from 4.3.12. If K is a knot with Seifert form

$$A_{-m} = \begin{pmatrix} 1 & 1 \\ 0 & -m \end{pmatrix},$$

and if m is positive, it follows from Levine's characterization of knots with quadratic Alexander polynomial (later in this book seen as Theorem 6.3.1) that K is of order two in the algebraic concordance group if every prime of odd exponent in $4m+1$ is congruent to 1 modulo 4. It follows as one example that any knot with Seifert form A_{-3}, for instance the -3–twisted double of the unknot, is of order 2 in algebraic concordance but is not concordant to a negative amphicheiral knot.

This gives insight into the following conjecture, based on a long standing question of Gordon [**?**]:

CONJECTURE 4.6.2. If K is of order two in $\mathcal{C}$, then K is concordant to a negative amphicheiral knot.

(Gordon's original question did not have the "negative" constraint in its statement.)

In a different direction, a knot is called strongly positive (negative) amphicheiral, if an involution of S^3 maps it to its mirror image and preserves (reverses) the string orientation. It was noted by Long [**?**] that the example of a knot K for which $K\#mK$ is not slice (described in the next subsection) yields an example of a nonslice strongly positive amphicheiral knot. Flapan [**?**] subsequently found a prime example of this type. It has since been shown that the concordance group contains infinitely many linearly independent such knots [**?**].

4.6.2. Reversibility and Mutation. [4]

In exercise 4.5.8 you saw that every twist knot is algebraically concordant to its reverse. The same holds for every knot. One proof follows from a stronger result of Long [**?**]: if K is strongly positive amphicheiral then it is algebraically slice. For any knot, $K\#mK$ is strongly positive amphicheiral, so K and rK are algebraically concordant. In Section 4.10 you will revisit reversibility and algebraic concordance with a few more tools. It is proved in [**?**] that there are knots that are not concordant to their reverses. Further examples have been developed in [**?, ?, ?**].

Kearton [**?**] observed that since $K\#mK$ is a (negative) mutant of the slice knot $K\#-K$, an example of a knot which is not concordant to its reverse also yields an example of mutation changing the concordance class of a knot. Similar examples for positive mutants proved harder to find and were developed in [**?, ?**].

4.6.3. Periodicity. A knot K is called periodic if it is invariant under a periodic transformation T of S^3 with the fixed point set of T a circle disjoint from K. Some of the strongest results concerning periodicity are those of Murasugi [**?**] constraining the Alexander polynomials of such knots. Naik [**?**] used Casson–Gordon invariants to obstruct

[4] Revisit this once we decide what to do with reverse and alg concordance. Should it just be moved?

periodicity for knots for which all algebraic invariants coincided with those of a periodic knot.

A theory of periodic, also known as *equivariant*, concordance has been developed. Basic results in the subject include those in [**?**, **?**] obstructing knots from being periodically slice and [**?**] giving a characterization of the Alexander polynomials of periodically ribbon knots.

4.6.4. Fibering. A knot is called fibered if its complement is a surface bundle over S^1. It is relatively easy to see that not all knots are concordant to fibered knots, as follows. The Alexander polynomial of a fibered knot is monic. Consider a knot K with $\Delta_K(t) = 2t^2 - 3t + 2$. If K were concordant to a fibered knot, then $\Delta_K(t)g(t) = f(t)f(t^{-1})$ for some monic polynomial g and integral f. However, since $\Delta_K(t)$ is irreducible and symmetric, it would have to be a factor of $f(t)$ and of $f(t^{-1})$, giving it even exponent in $\Delta_K(t)g(t)$, implying it is a factor of $g(t)$, contradicting monotonicity.

Soma [**?**] proved that fibered knots are concordant to hyperbolic fibered knots.

The most significant result associating fibering and concordance is the theorem of Casson and Gordon [**?**], stated below.

THEOREM 4.6.3. If K is a fibered ribbon knot, then the monodromy of the fibration extends over some solid handlebody.

4.7. The rational algebraic concordance group

Unlike the smooth or the topological knot concordance groups $\mathcal{C}$ and $\mathcal{C}_{top}$, the algebraic concordance group $\mathcal{G}^{\mathbb{Z}}$ is well understood. Recall Exercise 4.3.17, where we showed that there is a $(\oplus_1^\infty \mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/2\mathbb{Z})$ summand of $\mathcal{C}$. Prior exercises that led to that statement made use only of Seifert matrices. Therefore the same argument applies to show that $\mathcal{G}$ contains a summand isomorphic to $(\oplus_1^\infty \mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/2\mathbb{Z})$.

By studying the image of $\mathcal{G}^{\mathbb{Z}}$ into a Witt group of rational isometric structures and then obtaining a complete set of invariants using Milnor's work in [**?**], in [**?**, **?**] J. Levine proved that the rational algebraic concordance group $\mathcal{G}^{\mathbb{Q}}$ is isomorphic to $(\oplus_1^\infty \mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/2\mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/4\mathbb{Z})$. Furthermore, Levine showed that the algebraic concordance group $\mathcal{G}^{\mathbb{Z}}$ maps injectively into $\mathcal{G}^{\mathbb{Q}}$ and by constructing infinite, linearly independent families of algebraic concordance order of 2, 4 and ∞, respectively, he proved that

$$\mathcal{G}^{\mathbb{Z}} \cong (\oplus_1^\infty \mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/2\mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/4\mathbb{Z}).$$

Our goal is to develop algebraic tools used in proving this isomorphism. We follow this program in the next two chapters beginning with a general discussion of Witt groups and p-adic extensions of $\mathbb{Q}$. But first, let's study the image of $\mathcal{G}^{\mathbb{Z}}$ in $\mathcal{G}^{\mathbb{Q}}$.

EXERCISE 4.7.1. Show that the natural inclusion $\mathcal{G}^{\mathbb{Z}} \to \mathcal{G}^{\mathbb{Q}}$, obtained by sending the integral algebraic concordance class of A to the rational algebraic concordance class of A, is a well-defined group homomorphism.

Next, we want to establish injectivity. To this end, we need Theorem 4.7.2 and Exercise 4.7.3 below. Let $\mathcal{G}^{\mathbb{F}}$ be as defined in 4.5.7.

THEOREM 4.7.2. Every class in $\mathcal{G}^{\mathbb{F}}$ has a nonsingular representative.

Proof If a $2n \times 2n$ matrix A does not have maximal rank, i.e., if the number of linearly independent rows is less that $2n$, then row operations yield a matrix with a bottom row having 0 entries. Performing the corresponding column operations yields a matrix of the form

$$\begin{pmatrix} A_1 & b \\ 0 & 0 \end{pmatrix},$$

where A_1 is $(2n-1) \times (2n-1)$, b is $(2n-1) \times 1$, and the bottom row consists of 0 entries.

If A is a Seifert matrix, then $A - A^T$ is invertible and therefore some entry of b is nonzero. So a series of row operations, followed by a series of the corresponding column operations puts the matrix in the form

$$\begin{pmatrix} A_2 & b_1 & 0 \\ b_2 & b_3 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

where now A_2 is $(2n-2) \times (2n-2)$, b_1 is $(2n-2) \times 1$, b_2 is $1 \times (2n-2)$ and b_3 is 1×1. Further column operations, followed by row operations yield

$$\begin{pmatrix} A_2 & b_1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We want to see that this matrix is algebraically concordant (Witt equivalent) to A_2. We will achieve this by showing that the direct sum of $-A_2$ with this matrix is algebraically slice. The direct sum is of the form

$$\begin{pmatrix} C & b_1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

where $C = A_2 \oplus -A_2$ and b_1 the same as before. Now, however, we have that C is algebraically slice. A metabolizer for this form is built from a metabolizer for C by adding in the $2n$-dimensional vector with the last entry 1 and all other entries 0. Thus we have shown that a $2n \times 2n$ Seifert matrix with determinant zero is Witt equivalent to a $(2n-2) \times (2n-2)$ Seifert matrix. For a nontrivial knot this process has to terminate resulting in a nonsingular Seifert matrix in the algebraic concordance class. (The unknot may be represented by the 0×0 dimensional empty matrix.) $\square$

EXERCISE 4.7.3. Show that the above proof holds in the setting of integral matrices. That is, prove that an algebraic concordance class in $\mathcal{G}^{\mathbb{Z}}$ has a nonsingular integral Seifert matrix representative.

THEOREM 4.7.4. The natural inclusion $\mathcal{G}^{\mathbb{Z}} \to \mathcal{G}^{\mathbb{Q}}$ is an injective homomorphism.

Proof That, inclusion induces a well-defined homomorphism follows from Exercise 4.7.1. To see injectivity, let A be a $2n \times 2n$ non-singular integer Seifert matrix which represents $0 \in \mathcal{G}^{\mathbb{Q}}$. Let H be a rational metabolizer for A. We need to show that A has a metabolizer over $\mathbb{Z}$.

Viewing $V = \mathbb{Z}^{2n}$ as a subset of $\mathbb{Q}^{2n}$, let $H_0 = (H \cap V) \subset V$. Since A vanishes on $H \times H$, it vanishes on $H_0 \times H_0$. We wish to establish that H_0 is an n dimensional summand of V.

Suppose $\{h_1, \cdots, h_n\}$ is a basis for H over $\mathbb{Q}$. Choose positive integers a_i such that $a_i h_i \in H_0$. For example, $a_i =$ the least common multiple of denominators for all nonzero entries in h_i. Since h_i are linearly independent over $\mathbb{Q}$, $a_i h_i$ are linearly independent over $\mathbb{Z}$ and therefore the rank of H_0 over $\mathbb{Z}$ is at least n. Since $H_0 \otimes_{\mathbb{Z}} \mathbb{Q} \subset H \cong \mathbb{Q}^n$, the rank of H_0 is at most n. So $\mathrm{rk}(H_0) = n$ and $H_0 \cong \mathbb{Z}^n$. It remains to show that H_0 is a direct summand of $V \cong \mathbb{Z}^{2n}$.

Let $v \in V$ and suppose that for some non-zero integer m, we have $mv \in H_0 \subset H$. Since H is a rational vector space, $v = \frac{1}{m}(mv) \in H \implies v \in H \cap V = H_0$. It follows that the quotient space V/H_0 is torsion-free and so the exact sequence

$$0 \to H_0 \xrightarrow{i} V \to V/H_0 \to 0$$

is split exact. Therefore H_0 is a direct summand of V. □

We can now view $\mathcal{G}^{\mathbb{Z}}$ as a subgroup of $\mathcal{G}^{\mathbb{Q}}$. Our next goal is to follow Levine's approach in [?] to classify elements of $\mathcal{G}^{\mathbb{Q}}$ and then use Livingston's work in [?] to obtain results specific to the algebraic concordance group $\mathcal{G}^{\mathbb{Z}}$. Another approach to a complete classification of $\mathcal{G}^{\mathbb{Z}}$ was presented by Stoltzfus in [?].

As it is far easier to work with symmetric matrices than with general Seifert matrices, in the next section we study isometric structures which provide the translation from one setting to the other. A more algebraic treatment of related Witt groups is found in Chapter 5; in Section 5.6 we take this further to allow for skew symmetric forms that respect an involution.

4.8. Isometric structures and the Witt group $\mathcal{G}_{\mathbb{Q}}$

As before, let $\mathbb{F}$ be a field with characteristic not equal to two.

Definition 4.8.1. An isometric structure over a field $\mathbb{F}$ is a triple (V, B, S) where V is a $2n$–dimensional vector space, B is a nonsingular, symmetric, bilinear form on V and S is an isometry of V with respect to B, which means $B(x, y) = B(Sx, Sy)$, for all $x, y \in V$. Such an isometric structure is called admissible if the characteristic polynomial of S, namely, $\Delta_S(t) = \det(S - tI)$, satisfies the condition $\Delta_S(1)\Delta_S(-1) \neq 0$.

If we fix a basis of $V \cong \mathbb{F}^{2n}$ and write elements of V as column vectors,then S is represented by a non-singular (nonzero determinant) matrix and $B(x)$ is the matrix product Bx; B is represented by a symmetric matrix and $B(x, y)$ is the matrix product x^TBy, where x^T denotes the transpose of x.

Exercise 4.8.2. With B and S as in Definition 4.8.1, show that, in terms of matrices, we have $B = S^TBS$.

Exercise 4.8.3. Show that admissibility of the isometric structure (V, B, S) implies non-singularity of $I + S$.

Exercise 4.8.4. Show that $(I + S)^{-1} + (I + S^{-1})^{-1} = I$.

Exercise 4.8.5. Show that $(I + S)(I + S^{-1})^{-1} = S$.

Exercise 4.8.6. Show that $A = B(I + S)^{-1}$ is a nonsingular $\mathbb{F}$-Seifert matrix as defined in 4.5.1. **Hint:** As (1) $S^TBS = B \Leftrightarrow B^{-1}S^T = S^{-1}B^{-1}$ and (2) $B^T = B$, a simple matrix manipulation yields

$$A \pm A^T = B(I+S)^{-1} \pm (B(I+S)^{-1})^T = B\left((I+S)^{-1} \pm (I+S^{-1})^{-1}\right).$$

Now, use the above exercises to analyze $(I+S)^{-1} \pm (I+S^{-1})^{-1}$.

EXERCISE 4.8.7. If A is a $2n \times 2n$ dimensional, nonsingular $\mathbb{F}$-Seifert matrix, then $S = A^{-1}A^T$ is a linear transformation of the vector space $\mathbb{F}^{2n}$. Show that the characteristic polynomial $\Delta_S(t) = \det(S - tI)$ of the transformation S equals $\det(A)\Delta_A(t)$, where $\Delta_A(t) = \det(A - tA^T)$ is the standard Alexander polynomial.

(Note that in [**?**] Levine uses the linear transformation $-A^{-1}A^T$; we change signs to be consistent with the standard definition of the Alexander polynomial.)

Convention When working with isometric structures (V, B, S), we will necessarily work with $\Delta_S(t)$. When presenting results that can be applied directly to knots, we will, when possible, work with $\Delta_A(t)$.

EXERCISE 4.8.8. Let A be a $2n \times 2n$ dimensional, nonsingular $\mathbb{F}$–Seifert matrix. Show that $(\mathbb{F}^{2n}, B = A + A^T, S = A^{-1}A^T)$ is an admissible isometric structure as defined in 4.8.1.

EXERCISE 4.8.9. With A, B and S as in Exercise 4.8.8, show that the congruence class of A determines the congruence class of the bilinear form B and the similarity class of the linear transformation S. (A matrix M is said to be congruent to a matrix N if there is an invertible matrix P such that $N = PMP^T$, and M is similar to N if there is an invertible matrix P such that $N = PMP^{-1}$. In this context P corresponds to a change of basis for the underlying vector space.)

Next we wish to define a binary operation on isometric structures. To this end, consider isometric structures (V, B_V, S_V) and (W, B_W, S_W), where V and W are $\mathbb{F}$ vector spaces of dimension $2n$ and $2m$, respectively. We define

$$(V, B_V, S_V) \oplus (W, B_W, S_W) \;=\; (V \oplus W, B_V \perp B_W, S_V \oplus S_W),$$

where the direct sum of isometries $S_V \oplus S_W \colon V \oplus W \to V \oplus W$ is given by

$$(S_V \oplus S_W)(v \oplus w) = S_V(v) \oplus S_W(w)$$

and the orthogonal sum of forms $B_V \perp B_W \colon (V \oplus W) \times (V \oplus W) \to \mathbb{F}$, is given by

$$(B_V \perp B_W)(v_1 \oplus w_1, v_2 \oplus w_2) = B_V(v_1, w_1) + B_W(v_2, w_2).$$

We see below that $S = S_V \oplus S_W$ is an isometry with respect to $B = B_V \perp B_W$:

$$\begin{aligned}
&B\left(S(v_1 \oplus w_1), S(v_2 \oplus w_2)\right) \\
&= B\left(S_V(v_1) \oplus S_W(w_1), S_V(v_2) \oplus S_W(w_2)\right) \\
&= B_V\left(S_V(v_1), S_V(v_2)\right) + B_W\left(S_W(w_1), S_W(w_2)\right) \\
&= B_V\left(v_1, v_2\right) + B_W\left(w_1, w_2\right) \\
&= B\left(v_1 \oplus w_1, v_2 \oplus w_2\right).
\end{aligned}$$

If we fix bases for V and W, then B is the block sum or diagonal sum of matrices B_V and B_W and S is the block sum of S_V and S_W. We have

$$S(v \oplus w) \;=\; \begin{pmatrix} S_V & 0 \\ 0 & S_W \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \;=\; \begin{pmatrix} S_V \cdot v \\ S_W \cdot w \end{pmatrix}$$

and

$$B(v_1 \oplus w_1, v_2 \oplus w_2) \;=\; \begin{pmatrix} v_1^T & w_1^T \end{pmatrix} \begin{pmatrix} B_V & 0 \\ 0 & B_W \end{pmatrix} \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} \;=\; v_1^T \cdot B_V \cdot v_2 \;+\; w_1^T \cdot B_W \cdot w_2.$$

Our next goal is to define an equivalence relation on isometric structures and show that the addition defined above induces a group structure on the set of equivalence classes.

Definition 4.8.10. An isometric structure (V, B, S) of dimension $2n$ is called metabolic or Witt trivial (or in Levine's notation *null-cobordant*), if there is an n-dimensional S-invariant subspace of V on which B vanishes.

There is a Witt cancellation theorem for isometric structures stated in Exercise 4.8.11 below, which can be proved in the same way as Lemma 4.5.5 for Seifert forms.

Exercise 4.8.11. Prove that if $(V_1, B_1, S_1) \oplus (V_2, B_2, S_2)$ is Witt trivial and (V_2, B_2, S_2) is Witt trivial, then (V_1, B_1, S_1) is Witt trivial.

We can now define Witt equivalence (or cobordism) of isometric structures by setting (V_1, B_1, S_1) Witt equivalent to (V_2, B_2, S_2) if $(V_1, B_1, S_1) \oplus (V_2, -B_2, S_2)$ is Witt trivial.

Exercise 4.8.12. Prove that the set of Witt classes of isometric structures on $\mathbb{F}$ vector spaces with the group structure induced by direct sum is an abelian group.

Definition 4.8.13. The abelian group of Witt classes of isometric structures with the binary operation given by direct sums, as in 4.8.12, is denoted $\mathcal{G}_{\mathbb{F}}$.

We now describe the transition from Seifert matrices to isometric structures. Recall that by 4.7.2 and 4.7.3 classes in $\mathcal{G}^{\mathbb{Q}}$ and $\mathcal{G}^{\mathbb{Z}}$ have nonsingular representatives.

EXERCISE 4.8.14. Show that the map that sends a nonsingular $2n\times 2n$ $\mathbb{F}$-Seifert matrix A to the isometric structure $(\mathbb{F}^{2n}, A+A^T, A^{-1}A^T)$ induces a well-defined homomorphism η_1 from the Witt group $\mathcal{G}^{\mathbb{F}}$ to the Witt group $\mathcal{G}_{\mathbb{F}}$.

EXERCISE 4.8.15. Show that the map η_2, which sends the class of (V, B, S) to the class of $B(I+S)^{-1}$ is a well-defined homomorphism from $\mathcal{G}_{\mathbb{F}}$ to $\mathcal{G}^{\mathbb{F}}$.

THEOREM 4.8.16. The Witt groups $\mathcal{G}^{\mathbb{F}}$ and $\mathcal{G}_{\mathbb{F}}$ are isomorphic via $\eta_1\colon\, \mathcal{G}^{\mathbb{F}} \to \mathcal{G}_{\mathbb{F}}$ induced by sending a $2n \times 2n$ nonsingular $\mathbb{F}$-Seifert matrix A to the isometric structure $(\mathbb{F}^{2n}, A+A^T, A^{-1}A^T)$. We have $\eta_1^{-1} = \eta_2$, where η_2 is as defined in Exercise 4.8.15.

Proof By Exercise 4.8.14, η_1 is a well-defined homomorphism. We want to show that the homomorphism $\eta_2\colon\, \mathcal{G}_{\mathbb{F}} \to \mathcal{G}^{\mathbb{F}}$, defined in Exercise 4.8.15, is the inverse of η_1.

First note that $\eta_2\eta_1$ carries the class of A to the class of

$$(A+A^T)(I+A^{-1}A^T)^{-1} = A(I+A^{-1}A^T)(I+A^{-1}A^T)^{-1} = A.$$

To see that $\eta_1\eta_2$ is the identity, note that this composition carries (V, B, S) to

$$(V, B(I+S)^{-1} + (B(I+S)^{-1})^T, (B(I+S)^{-1})^{-1}(B(I+S)^{-1})^T.$$

As in the hint for Exercise 4.8.6, , it is easy to check that

$$B(I+S)^{-1} + (B(I+S)^{-1})^T = B\left((I+S)^{-1} + (I+S^{-1})^{-1}\right).$$

Now, from Exercise 4.8.4 it follows that $\eta_1\eta_2$ carries B to itself.

Next, $(B(I+S)^{-1})^T$ may be written as:

$$[\left((I+S)B^{-1}\right)^{-1}]^T = [B^{-1} + B^{-1}S^T]^{-1} = [B^{-1} + S^{-1}B^{-1}]^{-1} = B(I+S^{-1})^{-1}.$$

Therefore:

$$S \overset{\eta_1\eta_2}{\longmapsto} (B(I+S)^{-1})^{-1}(B(I+S)^{-1})^T = (I+S)B^{-1}B(I+S^{-1})^{-1}$$

By Exercise 4.8.5, the expression on the right hand side equals S. $\square$

Summarizing the work from the previous sections (see 3.3.5, 4.5.11, 4.7.4, and 4.8.16), we have

$$\mathcal{C} \overset{\text{onto}}{\to} \mathcal{C}_{top} \overset{\text{onto}}{\to} \mathcal{G}^{\mathbb{Z}} \overset{1-1}{\to} \mathcal{G}^{\mathbb{Q}} \overset{\cong}{\to} \mathcal{G}_{\mathbb{Q}}.$$

4.9. Decomposing $\mathcal{G}_{\mathbb{F}}$

To understand its structure and for computational purposes, it is beneficial to decompose $\mathcal{G}_{\mathbb{F}}$ into "smaller" pieces. This is done in Theorem 4.9.1 below. (See [**?**, Proposition 16].)

THEOREM 4.9.1. We have $\mathcal{G}_{\mathbb{F}} \cong \oplus_\delta \mathcal{G}^\delta_{\mathbb{F}}$ where $\mathcal{G}^\delta_{\mathbb{F}}$ is the subgroup of $\mathcal{G}_{\mathbb{F}}$ determined by isometric structures for which the characteristic polynomial of the isometry is a power of δ, and δ varies over all irreducible symmetric monic polynomials. In particular, a class $(V, B, S) \in \mathcal{G}_{\mathbb{F}}$ is trivial if and only if its projection $(V_\delta, B_\delta, S_\delta)$ in $\mathcal{G}^\delta_{\mathbb{F}}$ is trivial for all symmetric irreducible factors δ of the characteristic polynomial of S. Here $V_\delta = \operatorname{Ker}\left((\delta(t))^N\right)$, for large N, and B_δ and S_δ denote restrictions of B and S, respectively, to V_δ.

The proof of 4.9.1 is left to the reader, but most of the details are worked out below. Recall from Exercise 4.8.7 that if A is a nonsingular Seifert matrix, and $S = A^{-1}A^T$, then $\Delta_A(t) \doteq \det(A)\Delta_S(t)$. Similar to the results for Alexander polynomials of knots, we have:

THEOREM 4.9.2. Suppose that the characteristic polynomial $\Delta_S(t) = \det(S - tI)$ satisfies $\Delta_S(1)\Delta_S(-1) \neq 0$. Then:

(1) $\Delta_S(t) = t^{2n}\Delta_S(t^{-1})$.
(2) If $(V, B, S) = 0 \in \mathcal{G}_{\mathbb{F}}$ then $\Delta_S(t) = t^n f(t) f(t^{-1})$, where $f(t)$ is a polynomial of degree n.
(3) $\det(B) = \Delta_S(1)\Delta_S(-1) \in \mathbb{F}^*/(\mathbb{F}^*)^2$, where $\mathbb{F}^*$ denotes the nonzero elements in the field $\mathbb{F}$.

Now, for any polynomial $f(t)$ define $\bar{f}(t) = t^k f(t^{-1})$, with k chosen so that $\bar{f}$ is a polynomial with non-zero constant term. An irreducible polynomial f will be called non-symmetric if f and $\bar{f}$ have no non-trivial common factors. Otherwise it is called a symmetric polynomial. Since Δ_S is symmetric, if δ is a non-symmetric irreducible factor of Δ_S, then so is $\bar{\delta}$.

We can realize V as a module over the P.I.D. $\mathbb{F}[t, t^{-1}]$ by defining the action of t as $tv = S(v)$. Clearly V is annihilated by $\Delta_S(t)$, i.e., $\Delta_S(t)v = 0$, for each $v \in V$.

Consider the factorization of $\Delta_S(t)$ into powers of distinct irreducibles (primes). Define the δ-primary component of V as $V_\delta = \operatorname{Ker}\left((\delta(t))^N\right)$, for large N, where δ is an irreducible factor of Δ_S. Recalling the theory of finitely generated modules over principal ideal domains [**?**], we have $V \cong \oplus_\delta V_\delta$. The following two results tell us that it is sufficient to focus only on the V_δ corresponding to symmetric, irreducible factors of the Alexander polynomial. The first one is [**?**, Lemma 10].

LEMMA 4.9.3. Given irreducible factors δ and μ of Δ_S, either V_δ is orthogonal to V_μ, or $\delta \doteq \bar{\mu}$.

Proof Suppose that $\delta \not\doteq \bar{\mu}$, or equivalently δ and $\bar{\mu}$ are relatively prime, irreducible factors of Δ_S over $\mathbb{F}$. Let $u \in V_\delta$ and let $v \in V_\mu$. For a suffieciently large $N \in \mathbb{N}$ the linear transformation $\bar{\mu}(t)^N$, restricted to V_δ, has kernel 0. Therefore $\bar{\mu}(t)^N \colon V_\delta \to V_\delta$ is an isomorphism. It follows that each element u of V_δ can be written in the form $(\bar{\mu}(t))^N(w)$ for some $w \in V_\delta$. Now, since S is an isometry, we have $B(t^{-1}x, y) = B(x, ty)$, for any $x, y \in V$, and since $v \in V_\mu$, we have $\mu(t)^N v = 0$. It follows that

$$B\left(u,v\right) \,=\, B\left((\bar{\mu}(t))^N(w), v\right) \,=\, B\left(w, (\mu(t))^N v\right) \,=\, B\left(w,0\right) \,=\, 0.$$

So V_δ is orthogonal to V_μ. $\square$

The corollary below follows from the definition of a non-symmetric polynomial.

COROLLARY 4.9.4. If δ is a non-symmetric irreducible factor of Δ_S, the symmetric, bilinear form B restricted to $V_\delta \oplus V_{\bar{\delta}}$ vanishes on a subspace of half the dimension.

Proof Since δ is non-symmetric, it is relatively prime to $\bar{\delta}$. From 4.9.3 it follows that V_δ is orthogonal to itself. Thus, $V_\delta \oplus 0 \subset V_\delta \oplus V_{\bar{\delta}}$ is a subspace (metabolizer) of half the dimension of $V_\delta \oplus V_{\bar{\delta}}$ on which B vanishes. $\square$

The module V splits into the orthogonal sum of two types of submodules.

(1) V_δ, where δ is a symmetric irreducible factor of Δ_S, and
(2) $V_\delta \oplus V_{\bar{\delta}}$, where δ is a non-symmetric irreducible factor of Δ_S.

The restriction of (B, S) to submodules of each of these types gives an isometric structure. From 4.9.4 we know that ones of type (ii) are Witt trivial. Since the restrictions of B are nonsingular and any submodule of V is a direct sum of submodules of V_δ, it is clear that (V, B, S) is Witt trivial if and only if so is each $(V_\delta, B_\delta, S_\delta)$, where δ is a symmetric irreducible factor of Δ_S, and (B_δ, S_δ) is the restriction of (B, S) to V_δ. Also note that since $\Delta_S(t)$ is monic, we can choose each factor to be monic. In short, if $\Delta_S(t)$ factors as $\prod_i \delta_i(t)^{k_i} \prod_j g_j(t)^{l_j}$, where the δ_i are symmetric, monic, irreducible factors and the g_i are the non-symmetric irreducible factors, then let $\hat{\delta}_i = \Delta_S/\delta_i^{k_i}$, and note that $\hat{\delta}_i(t)^N$ is an isomorphism of V_{δ_i} and it annihilates all the other V_f summands if N is large. It follows that $V_{\delta_i} = \mathrm{Image}(\hat{\delta}_i^N(t))$ for any large N. In addition, S restricted to V_{δ_i} has characteristic polynomial $\delta_i^k(t)$ for some k, since δ_i was chosen to be monic.

EXERCISE 4.9.5. Prove Theorem 4.9.1.

Now we focus on $\mathcal{G}_{\mathbb{F}}^{\delta}$ and some special cases of δ.

THEOREM 4.9.6. An isometric structure (V, B, S) with $\Delta_S(t) = (\delta(t))^e$, where $\delta(t)$ is symmetric, irreducible, monic and $e > 0$, is either Witt equivalent to an isometric structure with minimal polynomial $\delta(t)$ or it is Witt trivial.

Proof Suppose that the minimal polynomial of S is $\delta(t)^a$, for some $a > 1$. It suffices to show that (V, B, S) is Witt equivalent to an isometric structure with minimal polynomial $\delta(t)^b$, for some $b < a$.

Consider the homomorphism $\phi\colon V \to V$ given by $\phi(v) = (\delta(t))^{a-1}(v)$. Since we have $t^{\pm 1} \cdot (\delta(t))^{a-1} v = \delta(t))^{a-1} \cdot t^{\pm 1} v$, it is a module homomorphism over the ring $\mathbb{F}[t, t^{-1}]$. The image $W = (\delta(t))^{a-1}(V)$ is a non-trivial submodule of V. Let $W^{\perp}$ denote the orthogonal complement of W; that is:

$$v \in W^{\perp} \Leftrightarrow B(v, \delta(t))^{a-1}(u)) = 0, \text{ for all } u \in V.$$

It is easy to see that $W^{\perp}$ is a submodule of V; in fact, since t is an isometry with respect to B and symmetry of δ implies $\bar{\delta} = \delta$, we have

$$0 = B(v, \delta(t))^{a-1}(u)) = B(\bar{\delta}(t))^{a-1}(v), u) = B(\delta(t))^{a-1}(v), u), \text{ for all } u \in V.$$

As B is nonsingular, this is equivalent to saying that $\delta(t))^{a-1}(v) = 0$. It follows that

$$W^{\perp} = \mathrm{Ker}(\phi).$$

Next, to see that $W \subset W^{\perp}$, note that for $v, w \in V$, we have

$$B((\delta(t))^{a-1} v, (\delta(t))^{a-1} w) = B(v, (\bar{\delta}(t))^{a-1} (\delta(t))^{a-1} w) = B(v, (\delta(t))^{2a-2} w) = 0,$$

since $2a - 2 \geq a$ for $a > 1$.

Now, consider the quotient module $W^{\perp}/W$ with the isometric structure, say (B', S'), induced by B and S. The minimal polynomial of S' divides $\delta(t)^{a-1}$, and so it is $\delta(t)^b$, where $b < a$.

We claim that (V, B, S) is Witt equivalent to $(W^{\perp}/W, B', S')$. This is seen by noting that the subspace

$$V_0 = \{ (v, [v]) \,|\, v \in W^{\perp} \text{ and } [v] \text{ is the class of } v \text{ in } W^{\perp}/W \}$$

is invariant under $S \oplus S'$ and $B \perp -B'$ vanishes on V_0. As $\dim(V_0) = \dim(W^\perp)$ and $\dim(V) = \dim(W) + \dim(W^\perp)$, we have $\dim(V_0) = \frac{1}{2}\left(\dim(V) + \dim(W^\perp/W)\right)$. Thus, V_0 is a metabolizer of $(V, B, S) \oplus (W^\perp/W, -B', S')$. □

When the characteristic polynomial is power of a quadratic (not necessarily irreducible), we have the stronger statement below.

THEOREM 4.9.7. An isometric structure (V, B, S) with $\Delta_S(t) = (\delta(t))^e$, where $\delta(t)$ is symmetric, quadratic, monic, but not necessarily irreducible, with $e > 0$, is Witt trivial if and only if the form B is Witt trivial.

Note that triviality of the form B in $W(\mathbb{F})$ corresponds to the existence of a half dimensional subspace over which B vanishes, or alternatively, to the existence of a basis for the underlying vector space such that B, with respect to this basis, is represented by a matrix with half-dimensional diagonal block of zeros. Witt groups $W(\mathbb{F})$ of symmetric bilinear forms are discussed in 5.1.

Proof As Witt triviality of the isometric structure implies that of the form B, the "only if" direction is automatic.

Next, if δ is reducible, then $\delta(t) = \beta(t)\bar{\beta}(t)$, V is a sum of $V_\beta \oplus V_{\bar{\beta}}$, and by Corollary 4.9.4, (V, B, S) is Witt trivial. So consider the case that δ is irreducible.

We write $\delta(t) = t^2 + at + 1$. By Theorem 4.9.6 we may assume that $\delta(t)$ is the minimal polynomial of S. (Otherwise (V, B, S) is Witt trivial, and we are done.) Now,

$$0 = B(\delta(S)v, Sv) = B(S^2v, Sv) + aB(Sv, Sv) + B(v, Sv) = 2B(Sv, v) + aB(v, v),$$

for any $v \in V$. Recalling that the field $\mathbb{F}$ does not have characteristic two, we have:

$$B(Sv, v) = \frac{-a}{2}B(v, v).$$

If B is Witt-trivial, there exists a nontrivial $v \in V$ such that $B(v, v) = 0$. Let W be the S-invariant subspace of V generated by this v.

From the above equation it follows that $W \subset W^\perp$. Now using Exercise 4.9.8 below and induction on the dimension of V the result follows.

□

EXERCISE 4.9.8. Show that the isometric structure in Theorem 4.9.7 is Witt equivalent to the induced structure on $W^\perp/W$, where, as defined in the proof of the theorem, W

is generated by a $v \in V$ such that $B(v,v) = 0$. (**Hint:** Consider V_0 as in the proof of Theorem 4.9.6.)

For the remainder of this section, let $\mathbb{F}$ be a field with characteristic 0. Our main interest is in the fields $\mathbb{R}$ or $\mathbb{Q}_p$, discussed more in Chapter 5.

If δ is an irreducible, symmetric polynomial, then $\mathbb{E} = \mathbb{F}[t,t^{-1}]/(\delta(t))$ is a field with involution induced by $t \mapsto t^{-1}$. In addition, if we have an $\mathbb{F}$-isometric structure (V,B,S) with such a $\delta(t)$ as the minimal polynomial for the isometry S, then V may be regarded as a vector space over $\mathbb{E}$ with the action of t given by S. By Lemma 1.1 of [**?**] there exists a unique, Hermitian, $\mathbb{E}$-form $[\ ]$ on V which makes the action of t an isometry and $B(x,y) = \mathrm{Trace}_{\mathbb{E}/\mathbb{F}}[x,y]$, for $x,y \in V$. The definitions of Witt groups of Hermitian forms and of isometric structures involving Hermitian forms closely follow those involving symmetric bilinear forms. The Witt triviality of (V,B,S) over $\mathbb{F}$ is equivalent to that of $[\ ]$ over $\mathbb{E}$, as seen below.

That, B vanishes on an $\mathbb{E}$-subspace (which is necessarily t-invariant) of V on which $[\ ]$ is 0, is obvious. Conversely, let V_0 be an S-invariant $\mathbb{F}$-subspace of V on which B vanishes. View V_0 as an $\mathbb{E}$-subspace and, if possible, suppose $x, y \in V_0$, such that $[x,y] = c \in \mathbb{E}$ and $c \neq 0$. We have $1 = c^{-1}[x,y] = [c^{-1}x, y]$. Since $\mathbb{F}$ has characteristic zero, $\mathrm{Trace}_{\mathbb{E}/\mathbb{F}}(1) \neq 0$. However, since c is represented by a polynomial in t, $tV_0 = S(V_0) = V_0$ and $x, y \in V_0$, we have $c^{-1}x, y \in V_0$. So $0 = B(c^{-1}x, y) = \mathrm{Trace}_{\mathbb{E}/\mathbb{F}}[c^{-1}x, y]$, which is a contradiction.

THEOREM 4.9.9. [**?**, Proposition 16] Let $\mathbb{F}$ be a field with characteristic 0 and let δ be an irreducible, symmetric polynomial over $\mathbb{F}$. A class $(V,B,S) \in \mathcal{G}^{\delta}_{\mathbb{F}}$ is trivial if and only if the characteristic polynomial of S equals an even power of δ and the form B is trivial in the Witt group $W(\mathbb{F})$ of $\mathbb{F}$.

Proof Suppose that $(V,B,S) \in \mathcal{G}^{\delta}_{\mathbb{F}}$ is trivial. Clearly, the form B is trivial in $W(\mathbb{F})$ and it follows from 4.9.2, (2) that the characteristic polynomial of S has to be an even power of δ.

Conversely, suppose that $\Delta_S(t) = (\delta(t))^{2e}$ and that B vanishes on a half dimensional subspace of V. Admissibility of the isometric structure (see 4.8.1) dictates that $\delta(t) \neq t \pm 1$. We will assume the minimal polynomial for S to be $\delta(t)$. Otherwise, by 4.9.6 it follows that (V,B,S) is Witt trivial and we are done.

As V is an even dimensional vector space over $\mathbb{E} = \mathbb{F}[t,t^{-1}]/(\delta(t))$, it supports a Witt trivial $\mathbb{E}$-Hermitian form $\langle\ \rangle$. For example, if $\{\alpha_1, \cdots, \alpha_e, \beta_1, \cdots, \beta_e\}$ is an $\mathbb{E}$-basis of V, define

$$\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0, \ \langle \alpha_i, \beta_j \rangle = \delta_{i,j}, \text{ for } 1 \le i, j \le e.$$

Now, $(V, \text{Trace}_{\mathbb{E}/\mathbb{F}}\langle\ \rangle, S)$ is an isometric structure and the subspace on which $\langle\ \rangle$ vanishes is a t-invariant metabolizer. (Action of t is via S.)

Finally, in [**?**, Theorem 2.1] Milnor proves that two isometric structures over $\mathbb{F}$ with congruent symmetric, bilinear forms and the same irreducible minimal polynomial are isomorphic.

The theorem now follows from the congruence of B and $\text{Trace}_{\mathbb{E}/\mathbb{F}} < >$. □

4.10. Reversibility and algebraic concordance again.

In exercise 4.5.8 you saw that the twist knots are all algebraically concordant to their reverse. As we mentioned in 4.6.2, it is a consequence of [**?**] that the same is true for all knots. The argument below will follow the logic of [**?**] for $K\# - K^r$ in particular.

First explain how the action of reversing a knot extends to $\mathcal{G}_{\mathbb{Q}}$.

Exercise 4.10.1. *The following diagram commutes, and every map is a homomorphism.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathcal{G}^{\mathbb{Z}} \arrow[r, "A \mapsto A^T"] \arrow[d] & \mathcal{G}^{\mathbb{Z}} \arrow[d] \\ \mathcal{G}_{\mathbb{Q}} \arrow[r, "{(V,B,S)\mapsto(V,B,S^{-1})}"] & \mathcal{G}_{\mathbb{Q}} \end{tikzcd}$$

Let $(U, C, T) = (V \oplus V, B \oplus -B, S \oplus S^{-1})$. The involution $\iota : U \to U$ defined by $\iota(u \oplus v) = v \oplus u$ will be key to the proof that $(U, C, T) = 0$ in $\mathcal{G}_{\mathbb{Q}}$.

Exercise 4.10.2.
- $\iota^2 = \text{Id}$
- *ι is an anti-isometry with repect to C, meaning $C(\iota(x), \iota(y)) = -C(x, y)$.*
- *ι has eigenvalues 1 and -1 with corresponding eigenspaces $U^+ = \{(x \oplus x) \mid x \in V\}$ and $U^- = \{(x \oplus -x) \mid x \in V\}$.*
- *C splits as a direct sum of U^+ and U^-. If $u \in U^+$ and $v \in U^-$ then $C(u, v) = 0$.*
- *For any $u \in U^+$ (or $u \in U^-$), the minimal T-invariant subspace containing u (denoted u_T) is isotropic[5] with respect to B, and it invariant with repect to ι.*

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Next, let L be a *maximal* isotropic subspace containing this u which is T and ι invariant. We claim is that L is a Lagrangian. We need only verify that $\dim(L) = \frac{1}{2}\dim(U)$.

EXERCISE 4.10.3.
- *If L is any ι-invariant space then $L = L^+ \oplus L^-$ where $L^+ = L \cap U^+$ and $L^- = L \cap U^-$.*
- *For $\epsilon \in \{+,-\}$, if $\dim(L^\epsilon) < \frac{1}{2}\dim(U^\epsilon)$ then there is some $x \in U^\epsilon$ for which $B(\ell, x) = 0$ for all $\ell \in L$.*
- *$L + x_T$ is isotropic and T and ι-invariant.*

Note that the result of the preceding exercise contradicts the maximality of L, so it must be that $\dim(L^\epsilon) = \frac{1}{2}\dim(U^\epsilon)$ for each $\epsilon \in \{+,-\}$, completing the proof that $(U, C, T) = (V \oplus V, B \oplus -B, S \oplus S^{-1})$ is metabolic.

CHAPTER 5

Witt class invariants

It is difficult to derive results concerning the algebraic concordance group working directly from its definition. Instead, one defines homomorphisms from $\mathcal{G}_{\mathbb{Q}}$ to Witt groups on which it is easier to define invariants.

In this chapter, after reviewing some of the basic theory of Witt groups in Section 5.1, we will begin by examining knot concordance invariants that arise from the Witt group $W(\mathbb{Q})$ of symmetric bilinear forms on $\mathbb{Q}$-vector spaces. These lead naturally to Witt groups over finite fields, $W(\mathbb{F}_p)$, which are considered in Section 5.2. In Section 5.3 we move on to much stronger and more subtle invariants arising from the Witt group of the field of rational functions, $W(\mathbb{Q}(t))$. Our main focus will be on the signature function defined on $W(\mathbb{Q}(t))$, but we will also indicate some of the number theoretic avenues that offer further insight into the structure of the group $\mathcal{G}_{\mathbb{Q}}$. Section 5.4 provides a brief introduction to p-adic completions of $\mathbb{Q}$. Section 5.5 introduces the Witt group of linking forms. We consider first the Witt group associated to forms defined on finite abelian groups and taking value in $\mathbb{Q}/\mathbb{Z}$. We then expand the discussion to consider forms on torsion $\mathbb{Q}[t, t^{-1}]$ modules. In both cases, the Witt groups are seen to be isomorphic to Witt groups discussed in the previous sections. The chapter concludes with Section 5.6, a discussion of Witt groups of isometric structures. An essential observation of this section is the connection between linking forms described in Section 5.5 and isometric structures. Here we outline the classification of the algebraic concordance group.

In Chapter 7 we will see how the constructions of this chapter arise naturally from the geometry of the knot, in particular from the cyclic covering spaces. Here however we will work entirely algebraically, beginning with a Seifert form or an associated Seifert matrix A.

Although the focus in this chapter is on $\mathcal{G}_{\mathbb{Q}}$, we will present examples to show that the invariants restrict nontrivially to the image of the algebraic concordance group $\mathcal{G}^{\mathbb{Z}}$. As the examples are all represented by small matrices, usually 2×2, it will be easy to pull back further (see 2.8.11) , and see the application of the invariants to explicit knots.

5.1. Witt groups

We begin with a brief discussion of Witt groups in general. Let R be a commutative ring with involution τ. That is, $\tau\colon R \to R$ is a homomorphism such that $\tau^2 = 1$. To simplify our algebra, we will assume that R is a PID. The main examples will be fields: rational numbers $\mathbb{Q}$ with trivial involution; subfields of complex numbers $\mathbb{C}$ with involution induced by complex conjugation; the field $\mathbb{Q}(t)$ of rational functions on $\mathbb{Q}$, with involution induced by the map that sends t to t^{-1}; and the finite fields of prime order $\mathbb{F}_p(= \mathbb{Z}/p)$, with trivial involution. We will also have occasion to consider the ring of integers $\mathbb{Z}$, and the polynomial ring $\mathbb{Q}[t, t^{-1}]$ with involution τ sending t to t^{-1}. Even these basic examples lead naturally to invariants arising from Witt groups over rings that are not PIDs, for example, rings of algebraic numbers. These will be discussed only briefly and as needed.

Given such a PID R with an involution τ, we consider nonsingular symmetric bilinear forms B on free finite rank R-modules, say V. Symmetry means that $B(x, y) = \tau(B(y, x))$; nonsingular means that the map $V \to \operatorname{Hom}(V, R)$ given by $x \to \phi_x$ with $\phi_x(y) = B(y, x)$ is a (skew linear) isomorphism; bilinear means that B is linear in the first variable and antilinear in the second variable, that is, $B(x, \alpha y) = \tau(\alpha)B(x, y)$. If τ is nontrivial, bilinearity is also refered to *Hermitian* or *sesquilinear*. Nonsingularity is equivalent to the matrix representation of B having a determinant which is a unit (i.e., invertible) in the ring. Once the underlying field (or ring) is fixed, it is customary to suppress a reference to the vector space (or module), and instead of (V, B) one tends to refer only to the form B.

Consistent with Definition 4.8.10, a symmetric, bilinear form B (or a pair (V, B)) is called *metabolic* or *Witt trivial* if there is a submodule H of V, with $\operatorname{rank}(V) = 2\operatorname{rank}(H)$, on which B is trivial. Forms B_1 and B_2 (or (V_1, B_1) and (V_2, B_2)) are considered equivalent if the direct sum $B_1 \oplus -B_2$ (equivalently, $(V_1 \oplus V_2, B_1 \oplus -B_2)$) is Witt trivial, and the set of equivalence classes is called the Witt group of R, denoted $W(R)$. Under the operation induced by direct sum, $W(R)$ forms an abelian group. (Compare 4.8.12.)

Two obvious issues now arise. The first is that the equivalence relation and resulting Witt group are well defined; this proof is quite similar to the arguments given in Sections 4.5, 4.8, and the reader should be able to provide it. The second issue is the description of the Witt groups for given rings. Results in this realm will be described as they arise in our consideration of the algebraic concordance group.

The basic theory of Witt groups of symmetric bilinear forms can be found in [**?, ?**]. For the most part we are interested only in forms over fields. In one case we need to consider $R = \mathbb{Z}_p$, the ring of p-adic integers, mentioned briefly in Section 4.8, and further discussed in 5.4. So we state the basic definitions in terms of a more general commutative ring R with unity and finitely generated free modules over R instead of vector spaces. To be more specific, let p be a prime, and let the ring R be either the field of rationals $\mathbb{Q}$, reals $\mathbb{R}$, p–adic numbers $\mathbb{Q}_p$, or the finite field with p elements $\mathbb{F}_p$, or the ring of integers $\mathbb{Z}$ or p-adic integers $\mathbb{Z}_p$.

If R is not of characteristic 2 (that is, any of the above rings under consideration except $\mathbb{F}_2$), any form B has a diagonal matrix representation with respect to some basis of V. In the case of $\mathbb{F}_2$, not every form is diagonalizable, but it is true that all Witt classes are represented by diagonal forms. We abbreviate such a diagonalization with the vector of its diagonal entries: $[d_1, \ldots, d_k]$. Notice that if B is diagonalized with respect to some basis and if a basis element is replaced with a multiple, then the corresponding diagonal entry is multiplied by the square of that constant.

We conclude this section with four basic observations regarding Witt groups:

Observation 5.1.1. The rank of a class $w \in W(R)$ is defined to be the rank of a form B that represents w. Unfortunately, it is not a Witt class invariant, as $rk \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2$; however, reduced modulo 2 it is. We denote that mod 2 reduction of the rank $rk_2 : W(R) \to \mathbb{Z}/2$. The kernel of rk_2 is called the fundamental ideal, $I(R)$, which consists of even rank forms.

Observation 5.1.2. The determinant of a form B is not a Witt class invariant. For example, $det \begin{pmatrix} 0 & b \\ \tau(b) & 0 \end{pmatrix} = -b\tau(b)$. However we define the *discriminant* of a form B by

$$disc(B) = (-1)^{r(r-1)/2} det(B) \in R^*_\tau / N(R^*) \subset R^*/N(R^*),$$

where r is the rank of B, R^* is the units of R, R^*_τ is the set of elements of R^* that are invariant under τ, and $N(R^*) = \{x\tau(x) | x \in R^*\} \subset R^*_\tau$, because τ is an involution. Clearly, If B is a form of rank $2g$, then $\mathrm{disc}(B) = (-1)^g \det(B)$. It is an easy exercise that $disc : I(R) \to R^*/N(R^*)$ is a well defined homomorphism. (In general, $R^*/N(R^*)$ is difficult to describe explicitly. For $R = \mathbb{F}_p$, a finite field, it is easily seen to be isomorphic to $\mathbb{Z}/2$ when $p > 2$. We will discuss a few other cases as they arise.)

OBSERVATION 5.1.3. The inclusion of a PID R into its field of fractions $\mathbb{F}$ induces an injection of Witt groups, $W(R) \to W(\mathbb{F})$. (Compare 4.7.4.)

OBSERVATION 5.1.4. For a field $\mathbb{F}$, every element of $W(\mathbb{F})$ is represented by a diagonal form; that is, as a direct sum of scalar forms. (If the characteristic of $\mathbb{F}$ is other than 2, this is easily shown to be true for forms. If the characteristic is 2, then it is only true in the Witt group, where the proof is an easy exercise.)

5.2. Concordance invariants from $W(\mathbb{Q})$

Given a rational Seifert form A, we can construct the nonsingular symmetric form $A + A^T$, and the map $A \to (A + A^T)$ yields a homomorphism $\mathcal{G}_{\mathbb{Q}} \to W(\mathbb{Q})$, with image contained in $I(R)$, the Witt group of even rank forms. Before examining the invariants of $W(\mathbb{Q})$, we illustrate that this map is not injective.

EXAMPLE 5.2.1. Consider the two Seifert forms

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & 7 \end{pmatrix}$$

Since the quotient of the determinants, $1/7$, is not a rational square, the forms are not algebraically concordant. However, the symmetrized forms, $A + A^T$ are equivalent over the rationals. For instance, diagonalizing the second, and then changing the diagonal entries by squares (corresponding to replacing a basis element with a multiple) yields the following sequence.

$$\begin{pmatrix} 2 & 1 \\ 1 & 14 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 0 & 27/2 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 0 & (2)(3) \end{pmatrix}$$

A similar sequence applied to the first form gives the same result.

5.2.1. The signature function. The simplest invariant of an element of $W(\mathbb{Q})$ is its signature σ, defined by diagonalizing a matrix representing the form and subtracting the number of negative entries on the diagonal from the number of positive entries. (See 2.10.9.) This is easily seen to be a homomorphism from $W(\mathbb{Q})$ to $\mathbb{Z}$. Composing the map $\mathcal{G}_{\mathbb{Q}} \to W(\mathbb{Q})$ with the signature yields a homomorphism, again denoted σ, in $\mathrm{Hom}(\mathcal{G}_{\mathbb{Q}}, \mathbb{Z})$. This homomorphism is called the *signature homomorphism*.

EXAMPLE 5.2.2. For the Seifert form A_m below, $\sigma(A_m) = 2$ if $m > 0$. Since σ is additive (see Theorem 2.10.11), A_m is of infinite order in $\mathcal{G}^{\mathbb{Z}}$, if $m > 0$. (Recall 4.3.12

and 4.6.1.)

$$A_m = \begin{pmatrix} 1 & 1 \\ 0 & m \end{pmatrix}$$

Observation 5.2.3. The signature homomorphism factors through the map induced by inclusion $W(\mathbb{Q}) \to W(\mathbb{R})$, and $W(\mathbb{R}) \cong \mathbb{Z}$

The first part is obvious. Any Witt class $\alpha \in W(\mathbb{R})$ has a diagonal representative $[1, \ldots, 1, -1, \ldots, -1]$. The sum of the entries is the signature, $\sigma(\alpha) \in 2\mathbb{Z}$. This induces an isomorphism $\sigma : W(\mathbb{R}) \to 2\mathbb{Z} \cong \mathbb{Z}$.

5.2.2. Invariants arising from $W(\mathbb{F}_p)$: In addition to the signature function, there are a number of easily computed invariants of $W(\mathbb{Q})$ which together yield a complete set of invariants. To begin with, for each prime integer p we define a homomorphism $\phi_p : W(\mathbb{Q}) \to W(\mathbb{F}_p)$, where $\mathbb{F}_p$ denotes the finite field with p elements.

Any form over a field is Witt equivalent to a diagonal form. We will write such a diagonal form in matrix notation as $\oplus_i(\alpha_i)$, where the α_i are nonzero elements of the field. Furthermore, in case of the field of rational numbers, we can arrange that each α_i is a square free integer.

The function $\phi_p : W(\mathbb{Q}) \to W(\mathbb{F}_p)$ is now defined by giving its value on generators: $\phi_p((\alpha))$. Write α as ap^k, where a is relatively prime to p. Set $\phi_p((\alpha)) = (a)$ if k is odd, and let $\phi_p((\alpha))$ equal the trivial element of $W(\mathbb{F}_p)$ if k is even.

It turns out that ϕ_p is a well defined homomorphism of Witt groups. A simple outline of the proof is as follows. First, as we mentioned above, the Witt group is generated by 1-dimensional forms, and next, the set of relations between such forms is generated by a simple set of relations between pairs of generators. It is then an easy matter to check that ϕ_p preserves these relations.

A deeper result states that combining these homomorphisms yields a surjective homomorphism $W(\mathbb{Q}) \to \oplus_p(W(\mathbb{F}_p))$, and that the kernel of this homomorphism is $W(\mathbb{Z})$. See [**?**]. Similar to the case of $W(\mathbb{R})$ above, it can be shown that $W(\mathbb{Z}) = \mathbb{Z}$, with the isomorphism given by the signature homomorphism.

Finally, it can be shown that $W(\mathbb{F}_p)$ is isomorphic to either $\mathbb{Z}/2$, $\mathbb{Z}/4$, or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ depending on whether $p = 2, 3$ or 1 mod 4, respectively. Here are some of the details in each of three possible cases.

OBSERVATION 5.2.4. $W(\mathbb{F}_2) \cong \mathbb{Z}/2$; the only invariant is the $\mathbb{Z}/2$ reduction of the rank.

Simultaneous row and column operations can reduce any form in $W(\mathbb{F}_2)$ to a direct sum of the forms represented by the matrices $\begin{pmatrix} 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. The first has order 2 in $W(\mathbb{F}_2)$ and the second is Witt trivial.

Next, if p is odd, let a be a non-square mod p. Any form in $W(\mathbb{F}_p)$ is equivalent to a diagonal form $[1, \ldots, 1, a, \ldots a]$.

OBSERVATION 5.2.5. For $p = 1 \bmod 4$, $W(\mathbb{F}_p) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$; the $\mathbb{Z}/2$ reduction of the rank and the discriminant give the isomorphism.

If $p = 1 \mod 4$ then -1 is a square mod p, so the discriminant is well defined on odd forms, and any diagonal form $[b, b]$ is equivalent to $[b, -b]$, which is Witt trivial, and hence all nontrivial forms are of order 2. Thus, every class is either trivial or one of $[1], [a]$, or $[1, a]$. Finally, it is easily checked that none of these are Witt equivalent to each other.

OBSERVATION 5.2.6. For $p = 3 \bmod 4$, $W(\mathbb{F}_p) \cong \mathbb{Z}/4$; a map to $\mathbb{Z}/4$ is defined as follows.

If $p = 3 \bmod 4$, then -1 is not a square, so a can be taken to be -1. The form $[1, -1]$ is Witt trivial, so every class is equivalent to a positive multiple of the diagonal form $[1]$ or the diagonal form $[-1]$. Any form $[b, b]$ is nontrivial, again since -1 is not a square mod p, but the form $[b, b, b, b]$ is trivial, with metabolizer $\langle (1, 0, a, b), (0, 1, b, -a) \rangle \subseteq \mathbb{F}_p^4$, where (a, b) satisfy $1 + a^2 + b^2 = 0$. (As described in the proof of [**?**, Lemma 3.3], the existence of such an (a, b) is implied by the *shoe box principle* . The set of values of x^2 contains $(p + 1)/2$ values in $\mathbb{F}_p$. Similarly, the set of values of $1 - y^2$ contains $(p + 1)/2$ values. Since the there are only p elements in $\mathbb{F}_p$, for some x and y, $x^2 = 1 - y^2$ has a solution.)

In short, when $p \equiv 3 \bmod 4$, even rank forms in $W(\mathbb{F}_p)$ map to 0 or $2 \in \mathbb{Z}/4$ depending on whether the discriminant is trivial or not. An odd rank form w maps to either 1 or 3 $\in \mathbb{Z}/4$ depending on whether or not the discriminant of $w \oplus (1)$ is trivial.

Composing these homomorphisms with the map of $\mathcal{G}_{\mathbb{Q}}$ to $W(\mathbb{Q})$ that carries A to $B = A + A^T$ yields a collection of invariants of $\mathcal{G}_{\mathbb{Q}}$.

EXAMPLE 5.2.7. Revisiting our favorite example of twisted doubles in 4.3.12, 4.6.1, 5.2.2, consider

$$A = A_m = \begin{pmatrix} 1 & 1 \\ 0 & m \end{pmatrix}.$$

Over the rationals $A + A^T$ can be diagonalized in a single step:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2m \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 0 & (4m-1)/2 \end{pmatrix} = B.$$

(1) For $p = 2$, $\phi_p(B) = 0 \in \mathbb{Z}/2$.
(2) For $p = 3 \bmod 4$, $\phi_p(B)$ is trivial if p has even exponent in $4a - 1$. If p has odd exponent in $(4m - 1)$, write $2(4m - 1)$ as bp^k where b and p are relatively prime; then $\phi_p(B) =< b >\in W(\mathbb{F}_p)$. In this case, $\phi_p(B)$ generates $\mathbb{Z}/4$.
(3) For $p = 1 \bmod 4$, $\phi_p(B)$ is trivial if p has even exponent in $4m - 1$. If p has odd exponent in $(4m - 1)$, again write $2(4m - 1)$ as bp^k where b and p are relatively prime; then $\phi_p(B) =< b >\in W(\mathbb{F}_p)$. In this case the rank invariant of $\phi_p(B)$ is nontrivial. The discriminant invariant is trivial or not, depending on whether b is a square or not.

5.3. Concordance invariants from $W(\mathbb{Q}(t))$

There is a significant generalization of the homomorphism described in the previous section. Later we will see how it derives from a geometric construction arising in knot theory, but for now we will present it in purely algebraic terms.

Given a Seifert form A, the associated form

$$B_t = [(1-t)A + (1-t^{-1})A^T]/2,$$

represents an element in the Witt group $W(\mathbb{Q}(t))$, where $\mathbb{Q}(t)$ denotes the rational functions in the variable t. (Notice the similarity with the matrix used to define the ω-signatures in 2.10.13.) It is easily shown that $\det(B_t) = (1-t)^k\Delta_A(t)$ for some k, and the determinant is nontrivial in $\mathbb{Q}(t)$. (Note that, evaluating B_t at $t = -1$ yields $A+A^T$, which by assumption has nontrivial determinant.) Hence, we have a well defined homomorphism from $\mathcal{G}_{\mathbb{Q}}$ to $W(\mathbb{Q}(t))$.

5.3.1. The signature function. There is a vast range of invariants defined on $W(\mathbb{Q}(t))$; once again, we begin by exploring the signature function. If the variable t in B_t is assigned a unit complex value, say ω, the form A_ω becomes hermitian, and hence has a well defined signature. If A is metabolic and A_ω is nonsingular, then this signature will be 0. In general the form can be singular (see Example 5.3.2 and Exercise 5.3.3), but only if ω is a root of the Alexander polynomial Δ_A or equal to 1. Hence, we can define an *averaged signature function*, denoted σ_ω by setting $\sigma_\omega(A)$ to be limit of the average value of the signatures of A_{ω_+} and A_{ω_-} where ω_+ and ω_- are points on the unit circle approaching ω from opposite sides, as in 2.10.16. We saw in 4.3.9 that the function σ_ω is a well defined homomorphism from the algebraic concordance group $\mathcal{G}^{\mathbb{Z}}$ to $\mathbb{Z}$.

In general, if an arbitrary representative of an element $W_t \in W(\mathbb{Q}(t))$ is selected, the same discussion applies, only now there can be a discrete set of points on the unit circle where the representative is singular, and another discrete set on which it is not well defined, corresponding to $0's$ of the denominators that appear in the representative. By averaging one again obtains a signature function that is well defined on $W(\mathbb{Q}(t))$.

EXAMPLE 5.3.1. Consider again the forms A_m from 5.2.7:

$$\begin{pmatrix} 1 & 1 \\ 0 & m \end{pmatrix}.$$

The associated form in $W(\mathbb{Q}(t))$ can be diagonalized to yield the form

$$1/2\begin{pmatrix} 1 & 0 \\ 0 & -mt + (2m-1) - mt^{-1} \end{pmatrix}.$$

Evaluated at $t = \omega$, the signature of this form is 2 or 0, depending on the sign of $2m(1 - \mathrm{Re}(\omega)) - 1$. That is, if $m > 1/4$, the signature is 2 for all ω near -1 and is 0 for all ω near 1; the signature function changes value only at $\mathrm{Re}(\omega) = 1 - 1/2m$. Hence, the set of these forms, as m ranges over the positive integers, forms a countable independent set in $\mathcal{G}^{\mathbb{Z}}$. (This completes Exercise 4.3.13.)

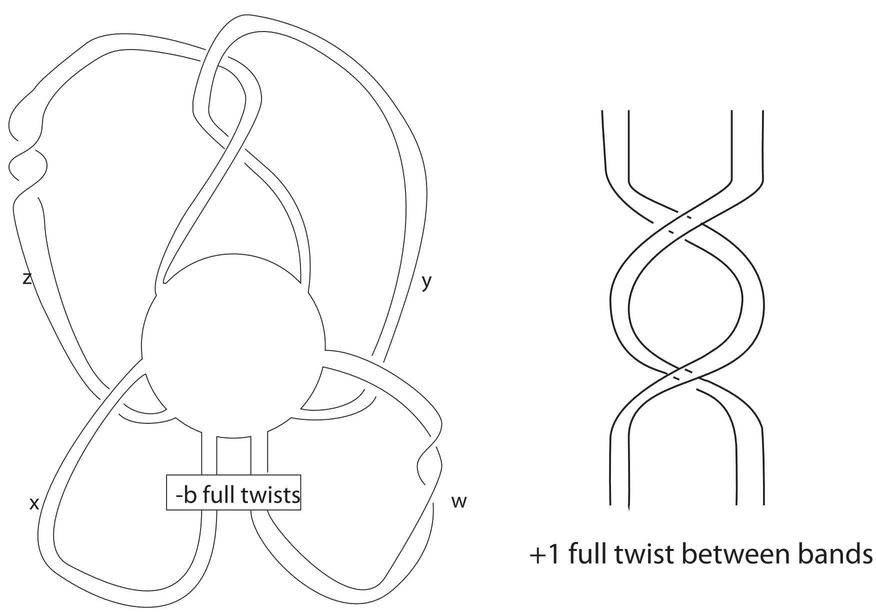
EXAMPLE 5.3.2. To more clearly understand the issue of the possible singularities of A_ω, consider the Seifert matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & -b \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -b & -1 & 0 & 1 \end{pmatrix}.$$

The determinant of B_t can be computed to be $(bt^2+(1-2b)t+b)^2(-1+t)^4/t^4$. The only 0's on the unit circle occur at ω_b, the unit complex number with real part $1-(1/2b)$.

Clearly then, away from ω_b the signature of A_ω is 0, since the form is metabolic. However, at ω_b the signature is 1. Computing this is a bit messy; it is easier to see that the signature is nonzero. If one restricts B_t to its lower diagonal 3×3 block, the determinant is $(t^2-3t+1)(1-t)^4/(8t^3)$, which is clearly nonsingular at ω_b if $b\neq -1$. Since the form is odd rank, the signature must be odd.

As outlined in the following Exercise 5.3.3, Figure 5.3.1 below shows a slice knot with this Seifert form, and hence it follows that slice knots can have nontrivial signature function.



EXERCISE 5.3.3. Let F be the genus-2 surface (disk with 4 bands) as shown in Figure 5.3.1. The 2 bands labeled x and w have b full negative twists around each other inside the box. The boundary of F is a knot K.

(1) Find appropriately oriented generating curves $x,\ y,\ z,\ w$ for $H_1(F)$, in the indicated bands, that give the Seifert matrix from the previous example.
(2) Prove that $K=\partial F$ is a slice knot. (Hint: Recall the "slice movie" in Figure 3.2 of Section 3.1. If each of the bands labeled z and w is pinched once, creating two saddle points, what is the resulting surface in the 4-ball, and what is its boundary?)

REMARK 5.3.4. As S-equivalence leaves the signature function unchanged, this function gives a well defined knot invariant. In light of this, we can use the previous example to make an interesting observation concerning double null concordance. If a knot is doubly null concordant, then its Seifert form is S-equivalent to a hyperbolic form, and its signature function would then be identically 0. Hence, the slice knots constructed in the previous example yield an infinite family of knots that are trivial in the knot concordance group, but are independent in the double concordance group.

5.3.2. Number fields and invariants from $W(\mathbb{Q}(\lambda))$**.** In Section 5.2.2, we extracted invariants from the Witt class of the form $A + A^T$ beyond the signature using finite fields—quotients of $\mathbb{Z}$ by prime ideals. Parallel to this, we will now explain how Witt class invariants can be extracted from the form $B_t = [(1-t)A + (1-t^{-1})A^T]/2$.

Before we constructed maps from the Witt group of rational numbers (the field of fractions of $\mathbb{Z}$) to $W(\mathbb{F}_p)$ for each prime p in $\mathbb{Z}$. Now we will describe homomorphisms from the Witt group of $\mathbb{Q}(t)$, the field of fractions of $\mathbb{Q}[t,t^{-1}]$, to the Witt group $W(\mathbb{Q}[t,t^{-1}]/ < p >)$, for each prime symmetric polynomial $p \in \mathbb{Q}[t,t^{-1}]$. (In most discussions one needs to consider prime ideals in the underlying domain, but since $\mathbb{Q}[t,t^{-1}]$ is a PID we need only look at irreducible polynomials.)

Let λ_p be a root of the polynomial p, so that we can write $\mathbb{Q}[t,t^{-1}]/ < p >$ as $\mathbb{Q}(\lambda_p)$. Then a map $\phi_p : W(\mathbb{Q}(t)) \to W(\mathbb{Q}(\lambda_p))$ is constructed as before. Diagonalize a given form so that the diagonal entries are all elements in $\mathbb{Q}[t,t^{-1}]$, with diagonal entries α_i, and factor $\alpha_i = p^k q_i$ where q_i is relatively prime to p. The homomorphism ϕ_p is defined by having it take the form (α) to the trivial Witt class if k is even, and to $(q(\lambda))$ if k is odd. Together these homomorphisms define a homomorphism from $W(\mathbb{Q}(t))$ to $\oplus_p W(\mathbb{Q}(\lambda_p))$, where the p range over all symmetric irreducible polynomials.

At this point, we have mapped $\mathcal{G}_{\mathbb{Q}}$ to $W(\mathbb{Q}(\lambda))$ for various λ. Explicit integer valued invariants can now be extracted by taking signature invariants of the forms in $W((\mathbb{Q}(\lambda)))$. Notice that each irreducible p may have several complex roots, and each leads to a different signature function. To go beyond these invariants, one must examine the detailed structure of $W(\mathbb{Q}(\lambda))$, and this leads deep into the realm of algebraic number theory. We will outline aspects of this next.

5.3.3. Rings of algebraic integers. The analysis of the Witt group $W(\mathbb{Q}(\lambda))$ extends into subtle number theory concerning the field $\mathbb{Q}(\lambda)$; here we will give a brief summary of how one proceeds. First, $\mathbb{Q}(\lambda)$ is the quotient field of an integral domain, $D(\lambda)$, defined to be the elements of $\mathbb{Q}(\lambda)$ that are roots of monic polynomials in $\mathbb{Z}[t]$. ($D(\lambda)$ does not equal $\mathbb{Z}(\lambda)$ in most cases, and in fact may not even contain λ.) Because $D(\lambda)$ is not in general a PID, the analog of the analysis of $\mathbb{Q}$ and $\mathbb{Z}$ via primes must be done at the level of prime ideals. (In $D(\lambda)$ the existence and uniqueness of prime decompositions persists, but only at the level of ideals.)

The consideration of a form in $W(\mathbb{Q}(\lambda))$ can now be done similarly to the rational case, again splitting the Witt group using primes. That is, we define a homomorphisms $\phi_p : W(\mathbb{Q}(\lambda)) \to W(D(\lambda)/p)$, for all prime ideals p in $D(\lambda)$. In fact, one need only consider the symmetric primes, those fixed by the involution induced by τ. Since any form can be diagonalized and denominators cleared, ϕ_p is defined for 1×1 forms, (a). If the prime ideal p appears to an even power in the prime factorization of the ideal generated by a, then $\phi_p((a))$ is defined to be trivial. On the other hand, if the ideal generated by a is of the form $b(p^k)$ with k odd, then we would like to define $\phi_p((a))$ to be the element $(b) \in W(D(\lambda)/p)$. If the ideal b is principal, this basically makes sense. If b is not principal, then an intermediate step must be added, based on the observation that b becomes principal in the ring formed by inverting all elements prime to p. The quotient $D(\lambda)/p$ is always a finite field, so an analysis much like that of the previous section of $W(\mathbb{F}_p)$ can be applied.

Finally we should note one further complication. In the case of rational forms, the kernel of the map $W(\mathbb{Q}) \to \oplus W(\mathbb{F}_p)$ is $W(\mathbb{Z})$, and that $W(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ via the signature homomorphism. In the present case, the kernel of the map $W(\mathbb{Q}(\lambda)) \to \oplus W(D(\lambda)/p)$ is $W(D(\lambda))$. As we mentioned before, this group has a number of signature functions mapping to $\mathbb{Z}$, one for each embedding of $D(\lambda)$ into $\mathbb{C}$. Taken together, these signatures (sometimes referred to as the *multisignature*) define a homomorphism of $W(D(\lambda))$ to a finitely generated free abelian group. In general there is a nontrivial kernel.

In practice, the procedure just described ranges from difficult to impossible. In the case that λ is quadratic or cyclotomic, the analysis of $D(\lambda)$, including its prime structure, is fairly well understood and is presented in texts on number theory. Other than this, little is known.

5.4. Real and p-adic completions of $\mathbb{Q}$

We should note that the procedure described above can be simplified to yield more accessible invariants. This is done by changing from rational to real coefficients. In this case the only irreducible polynomials are linear and degree two. It turns out that the linear factors do not contribute to the structure of the Witt group (compare 4.9.4), and the only quadratic factors that contribute have roots ω_a on the unit circle with real part $1-(1/2a)$ for some positive integer a. Hence, for each such a, there is a Witt class invariant coming from $W(\mathbb{R}(\omega_a))$, and here a signature invariant is defined.

Finally, rather than switching to real coefficients, one can use other completions of the rationals, the p-adic rationals.

Recall that for a prime integer p, a p–adic integer is defined to be a formal sum $\sum_{i=0}^{\infty} a_i p^i$, where the a_i satisfy $0 \leq a_i < p$. The set of p–adic integers is denoted $\mathbb{Z}_p$. Addition and multiplication are defined formally, and with these operations $\mathbb{Z}_p$ forms a commutative ring with unity. There is a natural inclusion of $\mathbb{Z}$ into $\mathbb{Z}_p$. The set of units in $\mathbb{Z}_p$ are those numbers for which $a_0 \neq 0$. Up to multiplication by a unit, p is the unique prime in $\mathbb{Z}_p$. The p–adic rationals, $\mathbb{Q}_p$, are defined to be formal sums $\sum_{i=k}^{\infty} a_i p^i$ where a_i satisfies $0 \leq a_i < p$, and k may be negative. Since any element in $\mathbb{Q}_p$ can be written as $p^i a$ where a is a unit in $\mathbb{Z}_p$, it is clear that $\mathbb{Q}_p$ is a field, and since it is the minimal field that contains $1/p$, it is the field for fractions of $\mathbb{Z}_p$. These are non-archemedian completions of $\mathbb{Q}$.

A good concise introduction to p–adic arithmetic is contained in the text by Serre [?]. Other references for the relevant algebraic number theoretic results include [**?, ?, ?, ?**].

Let $\mathbb{Q}_p^*$ denote the nonzero elements in $\mathbb{Q}_p$.

The structure of $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$

THEOREM 5.4.1. For p odd, the quotient of the multiplicative subgroup of nonzero elements of $\mathbb{Q}_p$ by its squares, $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$, is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Four distinct elements are given by the set $S = \{1, u, p, up\} \subset \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$, where u is any integer $0 < u < p$ which is not a square modulo p.

Proof Let $\mathbb{F}_p$ denote the field with p elements, also written as $\mathbb{Z}/p\mathbb{Z}$. Its multiplicative subgroup, $\mathbb{F}_p^*$, is a cyclic group of even order $p-1$. It follows that $\mathbb{F}_p^*/(\mathbb{F}_p^*)^2 = \mathbb{Z}/2\mathbb{Z}$. Thus, there is an element u that is not a square.

Clearly all elements of $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ are of order 2. It is easily seen that the set S in the statement of the theorem is a subgroup and that no product of any pair of distinct elements in S is a square, so S is a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Any element in $\mathbb{Q}_p^*$ can be multiplied by an even power of p so that it is of the form $s(a_0 + a_1 p + a_2 p^2 + \cdots)$, where $s \in S$ and a_0 is a square modulo p. Finally, a square root to $(a_0 + a_1 p + a_2 p^2 + \cdots)$ is easily found, using the fact the a_0 is a square modulo p and solving recursively for the coefficients. $\square$

The case of $p = 2$ is a bit more delicate, see below, and we leave the proof to [**?**].

THEOREM 5.4.2. The quotient of the multiplicative group of $\mathbb{Q}_2$ by its squares, $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$, is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Eight distinct elements are given by the set $S = \{\pm 1, \pm 2, \pm 5, \pm 10\} \subset \mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$.

DEFINITION 5.4.3. Any nonzero element $x \in \mathbb{Q}_p^*$ can be written as $p^\epsilon u$ where u is a unit in $\mathbb{Z}_p$. Set $D(x) \in \mathbb{Z}/2\mathbb{Z}$ be the mod 2 reduction of ϵ.

We note that the function D, defined above, descends to a function on $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$, which we denote $\bar{D}$.

Next, we want to delve a bit deeper into Witt groups building on observations from Section 5.2.2. Summarizing Observations 5.2.4, 5.2.5, and 5.2.6, we have the following.

THEOREM 5.4.4. A class in $W(\mathbb{F}_p)$ with p odd is uniquely determined by its mod 2 rank and discriminant. For $p = 2$ it is determined by its mod 2 rank.

OBSERVATION 5.4.5. For p odd, $W(\mathbb{Q}_p) \cong W(\mathbb{F}_p) \times W(\mathbb{F}_p)$

A form B over $\mathbb{Q}_p$ can be diagonalized as $[u_1, \ldots, u_k, pv_1, \ldots, pv_j]$ where the u_i and v_i are units in $\mathbb{Z}_p$[1]. If for a unit $u = a_0 + a_1 p + \cdots$ we let $\bar{u}$ denote the nonzero element $a_0 \in \mathbb{F}_p^*$, then we extract two forms in $W(\mathbb{F}_p)$: $[\bar{u_1}, \ldots, \bar{u_k}]$ and $[\bar{v_1}, \ldots, \bar{v_j}]$. This map provides the desired isomorphism. $\square$

Denote the isomorphism just defined by $\psi_p^e \oplus \psi_p^o$.

THEOREM 5.4.6. There is an exact sequence

$$0 \to W(\mathbb{Z}_p) \to W(\mathbb{Q}_p) \xrightarrow{\psi^o} W(\mathbb{F}_p) \to 0.$$

This is essentially Corollary 3.3 in Chapter 4 of [**?**]. The result there applies in the more general setting in which the last map need not be surjective. In our case surjectivity is clear.

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Observation 5.4.7. For $p = 2$, $W(\mathbb{Q}_2) \cong \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Here the generators are $[1], [-1, 5], \text{and}[-1, 2]$. (See [**?**, Chapter 5, Theorem 6.6].)

Observation 5.4.8. $W(\mathbb{Q}) \to \oplus_p W(\mathbb{F}_p)$

Any form in $B \in W(\mathbb{Q})$ can be diagonalized so that the diagonal entries are square free integers. Fix a prime p and write the diagonalized form as

$$[d_1, \ldots, d_k, pd_{k+1}, \ldots, pd_n],$$

where the d_i are all relatively prime to p.

The map $\psi_p^e : W(\mathbb{Q}) \to W(\mathbb{F}_p)$ sends B to $[d_{k+1}, \ldots, d_n]$. Combining these over all primes p gives a homomorphism $W(\mathbb{Q}) \to \oplus_p W(\mathbb{F}_p)$. □

See [**?**] for a proof that the kernel of the above homomorphism is $W(\mathbb{Z})$.

Theorem 5.4.9. If $(V, B) \in W(\mathbb{Q})$, p is a prime with $p \equiv 3 \mod 4$, and $\det(B) = p^r \frac{a}{b}$ with a and b relatively prime to p and r odd, then $\psi_p(B)$ has order 4 in $W(\mathbb{F}_p)$.

Proof $\psi_p(B)$ has odd rank. According to the analysis of $W(\mathbb{F}_p)$ for $p \equiv 3 \mod 4$ given in 5.2.6, if a form has odd rank, it is of order 4. □

Example 5.4.10. As a simple example, if one starts with the Seifert form for the knot $K(1, -5, 1)$ as in Figure 8.1, the Seifert matrix is

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -5 \end{pmatrix}, \quad \text{and} \quad A + A^T = \begin{pmatrix} 2 & 1 \\ 1 & -10 \end{pmatrix}.$$

Diagonalizing over the rationals yields

$$\begin{pmatrix} 2 & 0 \\ 0 & -(2)(3)(7) \end{pmatrix}.$$

With $p = 3$ this form maps to the element (-14) of $W(\mathbb{F}_3)$, which is equivalent to the form (1), a generator of order 4. The same is true working with $p = 7$.

Later, after Theorem 6.3.1, we will see that this particular form A is actually of order four in $\mathcal{G}$.

5.5. The Witt group of linking forms

5.5.1. $\mathbb{Q}/\mathbb{Z}$ linking forms. There is an alternative approach to the invariants described in Section 5.2 that arise via symmetric forms on $\mathbb{F}_p$ vector spaces. This alternative is based on the study of linking forms on finite abelian groups. In terms of Seifert forms the approach may seem less natural, but we will be seeing that linking forms arise more naturally in the setting of knot theory: linking forms are associated with 3-manifolds; intersection forms arise from 4- manifold considerations. We begin by describing the Witt theory of linking forms, and then show its relation to the previously defined $W(\mathbb{Q})$.

For the definition, let H be a finite abelian group; that is, H is a finite $\mathbb{Z}$ module. A *linking form* on H is a nonsingular symmetric bilinear form β on H taking values in $\mathbb{Q}/\mathbb{Z}$. Here, nonsingular means that the function from H to $\mathrm{Hom}(H, \mathbb{Q}/\mathbb{Z})$ induced by the pairing is an isomorphism.

A linking form is called *metabolic* if there is a subgroup $M \subset H$ with $|M|^2 = |H|$ and with β trivial when restricted to M. A Witt group of linking forms can be defined in the same way as for bilinear forms, and will be denoted $L(\mathbb{Z})$.

An easy exercise shows that a linking form splits as a direct sum over the p torsion subgroups of H for prime p, so one naturally defines the group $L(\mathbb{Z}, p)$ of linking forms on abelian p groups. (Here by p torsion we mean elements $a \in H$ for which $p^k a = 0$ for some k.) It follows that there is direct sum decomposition $L(\mathbb{Z}) = \oplus_p L(\mathbb{Z}, p)$.

There are two important results that we now describe. The first states that there is an homomorphism from $W(\mathbb{Q})$ to $L(\mathbb{Z})$. The next states that $L(\mathbb{Z}, p)$ is isomorphic to $W(\mathbb{F}_p)$. As a consequence, the invariants that arise from linking forms are the same as those that arise from $W(\mathbb{F}_p)$, as described in Section 3.

The homomorphism from $W(\mathbb{Q})$ to $L(\mathbb{Z})$ is defined as follows. Given a form $W \in W(\mathbb{Q})$ defined on the $\mathbb{Q}$-vector space H, choose a lattice $L \subset H$ on which W evaluates integrally. (A lattice is a maximal rank $\mathbb{Z}$-submodule of H.) Such an L is easy to find. Let $L^\#$ denote the set of elements in H which pair integrally against all elements in L; $L^\# = \{x \in H | W(x, y) \in \mathbb{Z} \ \forall \ y \in L\}$. It is easy to show that W induces a linking form on the finite quotient, $L^\#/L$. The following result is for the most part an exercise in linear algebra, the proof of which we do not include. The identification of the kernel is nontrivial.

THEOREM 5.5.1. The procedure described in the previous paragraph yields a well defined homomorphism from $W(\mathbb{Q})$ to $L(\mathbb{Z})$, with kernel $W(\mathbb{Z})$.

Remark 5.5.2. There is an important alternative description of the previous construction. A symmetric form B on a free $\mathbb{Z}$-module H determines a homomorphism $f : H \to \mathrm{Hom}(H, \mathbb{Z})$, with the property that $f(a)(b) = f(b)(a)$. Any such homomorphism determines a $\mathbb{Q}/\mathbb{Z}$ valued linking form, L, on the the torsion subgroup of $\mathrm{Hom}(H, \mathbb{Q})/f(H)$ as follows. Given torsion elements x and y of this quotient, mx is in $f(H)$, so we can write $mx = f(h)$. Define $L(x, y) = y(h)/m$. It is an easy exercise to check that this is well defined and corresponds to construction above. In the case that we have a $\mathbb{Q}$ valued bilinear form, one must restrict to a sublattice on which the form is integer valued, and then check that the choice of sublattice does not affect the outcome.

Example 5.5.3. Let B be a symmetric integral matrix representing an bilinear form on $\mathbb{Z}^n$. An obvious choice of lattice is $L = \mathbb{Z}^n$, in which case one checks that $L^\# = B^{-1}(L)$. The matrix B defines an isomorphism from $L^\#/L$ to $B(L^\#)/B(L) = \mathbb{Z}^n/B(\mathbb{Z}^n)$, a finite group of order $det(B)$. The standard basis of $\mathbb{Z}^n$ yields a generating set for the quotient; with respect to this generating set, B^{-1} represents the linking form, when viewed as a matrix with values in $\mathbb{Q}/\mathbb{Z}$.

Theorem 5.5.4. There is a natural isomorphism from $L(\mathbb{Z}, p)$ to $W(\mathbb{F}_p)$.

Proof The proof consists of showing that every linking form on a p group is Witt equivalent to a form on $(\mathbb{Z}/p)^k$ for some k. The isomorphism is then easily seen, since any bilinear function from such a group to $\mathbb{Q}/\mathbb{Z}$ can be viewed as a $\mathbb{F}_p$ valued pairing on a $\mathbb{F}_p$ vector space. The process of reducing H to a direct sum of copies of $\mathbb{Z}/p$ is called *devissage*, and will now be described.

Suppose that $p^k H \neq 0$ for some $k > 1$. Pick a maximal such k, and let $L = p^{k-1}H$. Since β can be viewed as taking values in $\mathbb{Z}/p^k$, it is clear that β vanishes on L. Let $L^\perp = \{x \in H | \beta(x, l) = 0\ \forall\ l \in L$. Then β induces a linking form β' on $L^\perp/L$.

The proof is completed by induction once it is noted that β' is Witt equivalent to β. To see this, a metabolizer must be found for $\beta \oplus -\beta'$ defined on $H \oplus L^\perp/L$. There is a diagonal injection of $L^\perp$ into this direct sum, and it is clear that the form vanishes on the image. To establish that the image is a metabolizer, all that remains to show is that $|L^\perp|^2 = |H||L^\perp|/|L|$. That is, that $|H| = |L||L^\perp|$.

For this last equality, note that there is a map of H onto $\mathrm{Hom}(L, \mathbb{Q}/\mathbb{Z})$ given by β. (To achieve surjectivity, use that the form is nonsingular on H. Also, the restriction $\mathrm{Hom}(H, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(L, \mathbb{Q}/\mathbb{Z})$ is surjective.) The kernel is exactly $L^\perp$.

5.5.2. $\mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}]$ **linking forms.** Almost identical to the way that one moves from rational symmetric forms to $\mathbb{Q}/\mathbb{Z}$-valued linking forms, there is a map from $W(\mathbb{Q}(t))$ to $L(\mathbb{Q}[t,t^{-1}])$. Similarly, there is a direct sum decomposition

$$L(\mathbb{Q}[t,t^{-1}]) = \oplus_p L(\mathbb{Q}[t,t^{-1}],p)$$

with p ranging over symmetric irreducible polynomials. (For an asymmetric p, the linking form is Witt trivial on the direct sum of the p and $\tau(p)$ torsion submodules.) And finally, there is an isomorphism from $L(\mathbb{Q}[t,t^{-1}],p)$ to $W(\mathbb{Q}[t,t^{-1}]/p) = W(\mathbb{Q}(\lambda_p))$, again via devissage.

Our previous observation that symmetric homomorphisms from H to $\mathrm{Hom}(H,\mathbb{Z})$ are related to linking forms applies in the present context as well, with the appropriate modification for the symmetry condition.

5.5.3. $\mathbb{R}(t)/\mathbb{R}[t,t^{-1}]$ **linking forms.** If we move from rational to real coefficients, we gain access to one of the important results relating concordance invariants; more specifically the signature function of B_t is determined by signature invariants associated to a linking form. This connection will later be seen to be tied to the relationship between the intersection form of a 4-manifold associated to a knot, and the linking form (or *Blanchfield pairing* of the infinite cyclic cover of the knot complement.

Over $\mathbb{R}(t)$ the form B_t can be diagonalized and denominators cleared, and each diagonal entry can be factored as $p_1^{k_1}\dots p_n^{k_n}q$ where $p_i(t) = t+a_i+1/t$ is an irreducible quadratic having roots ω_i on the unit circle and q is the product of the remaining irreducible factors, each linear or quadratic with nonunit roots.

In analyzing the signature function we can at this point consider the case of the 1-dimensional form with matrix representation $(\Delta) = (p_1^{k_1}..p_n^{k_n}q)$, as above. We note that the signature function has possible jumps at the roots of the p_i, and these occur only if k_i is odd. In this case, the jump of the signature is determined by the sign of $\Delta/p_i^{k_i}$ evaluated at the ω_i. (Compare Exercise 2.10.14.)

The linking form associated to the given form takes value in $\oplus_i L(\mathbb{R}[t,t^{-1}],p_i) = \oplus_i W(\mathbb{R}(\omega_i))$. There is a signature defined on $W(\mathbb{R}(\omega_i))$, and for rank 1 forms the signature is given by the sign of the entry. Examining the isomorphisms and devissage explicitly, one finds that for k_i even, the image of the form in $W(\mathbb{R}(\omega_i))$ is trivial, but when k_i is odd, the image is given by the rank 1 form with matrix representation $(\Delta/p_i^{k_i})$, evaluated at ω_i. Hence we have the following:

THEOREM 5.5.5. The signature function for B_t has jumps precisely at the unit roots ω of quadratics p, and these jumps are given by twice the signature of the associated form in $W(\mathbb{R}(\omega))$.

5.6. Isometric structures and the classification of $\mathcal{G}_{\mathbb{Q}}$

There is one last type of Witt group, discussed briefly in Section 4.8, that is of use in studying the algebraic concordance group $\mathcal{G}^{\mathbb{Z}}$ and is most easily used in the classification theorem. These are the Witt groups of isometric structures. There are a number of definitions of an isometric structure. In each case one begins with a triple (V, B, S), where V is a $\mathbb{F}$ vector space, B is an bilinear form on V (Hermitian when $\mathbb{F}$ has a nontrivial involution) and S is automorphism of V satisfying certain compatibility conditions with B. A form is called metabolic if there is an S-invariant subspace of V on which B vanishes. Note that in 4.8, B was required to be symmetric.

As our main example we consider the case where $\mathbb{F} = \mathbb{Q}$, B is skew symmetric and S is an isometry; that is $B(Sx, y) = B(x, S^{-1}y)$ for all x and y. Denoting the associated Witt group of isometric structures by $W_I(\mathbb{F})$ we will describe a map, Λ, from $\mathcal{G}_{\mathbb{Q}}$ to $W_I(\mathbb{F})$. We will briefly discuss the techniques used to classify $W_I(\mathbb{F})$ and relate the approach to Levine's original classification of $\mathcal{G}^{\mathbb{Z}}$. We will also relate this work to our previously defined invariants, in particular indicating how Λ factors through both $W(\mathbb{Q}(t))$ and $L(\mathbb{Q}(t))$, where the action of t is via the isometry S..

The definition of Λ is best stated in terms of matrices: if A is an element of $\mathcal{G}_{\mathbb{Q}}$, defined on $V = \mathbb{Q}^n$, then we pick a matrix representing A, which we again denote A, and consider the isometric structure $(V, A - A^T, A^{-1}A^T)$. Compare this with Section 4.8, where we used the symmetric, bilinear form $A + A^T$, as in Levine's original approach, in place of the skew-symmetric $A - A^T$.

THEOREM 5.6.1. The map $\Lambda : \mathcal{G}_{\mathbb{Q}} \to W_I(\mathbb{Q})$ is injective. It is onto the subgroup $W_I^0(\mathbb{Q}) \subset W_I(\mathbb{Q})$ spanned by isometric structures for which $S - 1$ is invertible.

Proof Since it is assumed that $A - A^T$ is invertible, we have immediately that $\Lambda(A)$ is in $W_I^0(\mathbb{Q})$. The matrix A may not be invertible, but as in Theorem 4.7.2, we see that every class in $\mathcal{G}_{\mathbb{Q}}$ has an invertible representative, and that the choice of that representative does not affect the Witt class of the resulting isometric structure. Given a form $(V, B, S) \in W_I^0(\mathbb{Q})$, using matrix representatives one checks easily that the function $(V, B, S) \to$

$BS^{-1}(S^{-1}-I)^{-1}$ defines an inverse to Λ. (Compare Exercise 4.8.15.) Hence we have both the injectivity of Λ and the surjectivity onto $W_I^0(\mathbb{Q})$. □

5.6.1. A summary of Levine's invariants and classification. In [**?**] Levine considered a slightly different group of isometric structures, assuming that B is symmetric and considering the map that takes A to $A+A^T$. Our choice was dictated by the fact that skew symmetric forms will arise more naturally in studying knots in dimension 3. Using the map Λ one can construct an isomorphism between the two Witt groups.

We now give a very brief summary of Levine's work on the Witt classification of (symmetric) isometric structures. There are three types of invariants which together determine the Witt class of an isometric structure:

(a) For each symmetric irreducible factor λ of the charactertic polynomial of t there is the exponent of λ mod 2, denoted ϵ_λ.
(b) Working over the real numbers, for each symmetric irreducible factor λ of the characteristic polynomial of t there is the signature, σ_λ of B restricted to the subspace of H annihilated by λ and its powers.
(c) Working over the p-adic completion of $\mathbb{Q}$ Levine defines a $\mathbb{Z}/2$ valued invariant μ_λ, again for each irreducible factor λ of the characteristic polynomial of t. (These are defined in terms of the Hasse symbol of B restricted to the summand of H annihilated by powers of λ. These invariants do not form homomorphisms, but their failure to be additive can be analyzed.)

Levine's main result [**?**, Proposition 22] states that the vanishing of obstructions provided by these invariants (over all mentioned completions and all symmetric irreducible polynomials) implies that the isometric structure is Witt trivial.

The invariants ϵ_λ

The following is an easy corollary of 4.3.2. (Also see Exercise 4.3.7.)

COROLLARY 5.6.2. If the Seifert matrix is algebraically slice and λ is a symmetric irreducible factor of the Alexander polynomial Δ, then the exponent of λ in Δ is even.

THEOREM 5.6.3. The functions ϵ_λ are distinct elements of $\mathrm{Hom}(\mathcal{G}_{\mathbb{Q}}, \mathbb{Z}_2)$.

Proof By the previous corollary, these are well defined on $\mathcal{G}_{\mathbb{Q}}$ and are clearly additive. That they are distinct is less clear. We have observed in 2.10.7 that every integral symmetric polynomial taking value ± 1 at 1 occurs as the polynomial of a knot. □

Milnor and Fox [?] used this to define a surjective homomorphism of $\mathcal{G}$ to $\mathbb{Z}_2^\infty$. The knots $K(a,-a,1)$ of Figure 8.1 are of order at most 2 in $\mathcal{C}$ since for each, $K(a,-a,1) = -K(a,-a,1)$. On the other hand, these have distinct irreducible Alexander polynomials if $a > 0$. The existence of an infinite summand of $\mathcal{G}$ isomorphic to $\mathbb{Z}_2^\infty$ follows. Note that the knot $K(1,-1,1)$ is the figure eight knot. (See Exercise 4.3.16.)

The signature invariants σ_λ

The symmetric, irreducible factors of Δ_A over $\mathbb{R}$ are quadratic and of the form $\lambda(t) = (t-z)(t-\bar{z})$, where $z = e^{i\theta}$ is a unit complex number. The invariants contributed by these are the signatures defined in 2.10.13. They are signatures of the bilinear form restricted to V_λ, the $\lambda(t)$-primary component of V.

The following result is [**?**, Proposition 22].

THEOREM 5.6.4. Suppose $\alpha \in \mathcal{G}_{\mathbb{Q}}$. Then

(1) The class α has finite order if and only if every $\sigma_\lambda(\alpha) = 0$.
(2) If α has finite order, then $4\alpha = 0$; therefore every element of $\mathcal{G}_{\mathbb{Q}}$ has order $1,\ 2,\ 4$ or ∞.
(3) The class α has order 4 if and only if all $\sigma_\lambda(\alpha) = 0$, but, for some $\lambda(t)$ over a non-archimedean completion $\mathbb{Q}_p$, $\epsilon(\lambda) \neq 0$, and $\big((-1)^d\lambda(1)\lambda(-1), -1\big) \neq 0$.

Here $(\ ,\ )$ denotes the Hilbert symbol.

Thus, we have:

$$\mathcal{G}^{\mathbb{Z}} \hookrightarrow \mathcal{G}_{\mathbb{Q}} \to (\oplus_1^\infty \mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/2\mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/4\mathbb{Z}).$$

In Exercise 4.3.17 we have already identified a $(\oplus_1^\infty \mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/2\mathbb{Z})$ summand within $\mathcal{C}$ that maps isomorphically into $\mathcal{G} = \mathcal{G}^{\mathbb{Z}}$. In Chapter 6 we show that, in fact:

$$\mathcal{G}^{\mathbb{Z}} \cong (\oplus_1^\infty \mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/2\mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/4\mathbb{Z}).$$

5.6.2. Relating the Witt groups:

$$\mathcal{G}_{\mathbb{Q}} \to W(\mathbb{Q}(t)) \to L(\mathbb{Q}(t)) \to W_I^0(\mathbb{Q}) \to \mathcal{G}_{\mathbb{Q}}$$

Only the middle function $\chi* : L(\mathbb{Q}(t)) \to W_I^0(\mathbb{Q})$ remains to be defined, and then we verify that the composition of this sequence of maps is the identity on $\mathcal{G}_{\mathbb{Q}}$.

To define $\chi*$ we begin by defining a map $\chi : \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}] \to \mathbb{Q}$. This is done as follows. Any element of $f \in \mathbb{Q}(t)$ can be expanded as a power series $\sum_{i=m}^{\infty} a_i t^i$ and also as a series $\sum_{i=m}^{\infty} b_i t^{-i}$. Set $\chi(f) = a_0 - b_0$. It is an elementary exercise to show that χ is well defined. A somewhat more difficult result states that χ induces a map of Witt groups, (where the isometry t is given by the action of t on the $\mathbb{Q}[t,t^{-1}]$ module) and that this induces an isomorphism of Witt groups.

5.6.3. Diagrams and summary. The results just discussed can be summarized with the following diagrams. In each case, the direct sum is taken over symmetric primes or prime ideals. Rows and columns are exact, and the right most vertical maps are isomorphisms.

The first diagram is relevant to the study of $A + A^T$, and we recall that $W(\mathbb{Z}) = \mathbb{Z}$ via the signature and $W(\mathbb{F}_p)$ is isomorphic to either $\mathbb{Z}/2$, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, or $\mathbb{Z}/4$.

The second two diagrams are relevant to the study of $[(1-t)A + (1-t^{-1})A^T]$. Here we presented less information, but noted that for each prime in $D[\lambda]$ the quotient $D[\lambda]/p$ is a finite field for which the analysis is fairly straightforward. The Witt group $W(D[\lambda])$ has signature invariants, but has further structure beyond this, and the Witt group $W(\mathbb{Q}[t,t^{-1}])$ can be proved to be isomorphic to $W(\mathbb{Q})$ via the natural inclusion. [2]

$$\left(\begin{array}{ccccc} & & W(\mathbb{Z}) & & \\ & & \downarrow & & \\ W(\mathbb{Z}) & \to & W(\mathbb{Q}) & \to & \oplus_p W(\mathbb{F}_p) \\ & & \downarrow & & \downarrow \\ & & L(\mathbb{Z}) & \to & \oplus_p L(\mathbb{Z}\mathbb{Z}, p) \end{array}\right)$$

$$\left(\begin{array}{ccccc} & & W(\mathbb{Q}[t,t^{-1}]) & & \\ & & \downarrow & & \\ W(\mathbb{Q}[t,t^{-1}]) & \to & W(\mathbb{Q}(t)) & \to & \oplus_p W(\mathbb{Q}[t,t^{-1}]/p) \cong \oplus_p W(\mathbb{Q}[\lambda_p]) \\ & & \downarrow & & \downarrow \\ & & L(\mathbb{Q}[t,t^{-1}]) & \to & \oplus_p L(\mathbb{Q}[t,t^{-1}], p) \end{array}\right)$$

[2]

$$\left(\; W(D[\lambda]) \;\; \to \;\; W(\mathbb{Q}(\lambda)) \;\; \to \;\; \oplus_p W(D[\lambda]/p) \; \right)$$

CHAPTER 6

Order in Algebraic Concordance

In Chapter 4, Theorem 4.5.9, we established a homomorphism from the concordance group $\mathcal{C}$ of classes of knots to the algebraic concordance group $\mathcal{G}$ of classes of Seifert matrices. Several algebraic results then culminated into an injection of $\mathcal{G}$ into the Witt group $\mathcal{G}_{\mathbb{Q}}$ of rational isometric structures (Theorems 4.7.4, 4.8.16), which we investigated further in Chapter 5. In Theorem 5.6.4, we obtained a homomorphism from $\mathcal{G}_{\mathbb{Q}}$ to $(\oplus_1^\infty \mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/2\mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/4\mathbb{Z})$. Exercise 4.3.17 illustrated a $(\oplus_1^\infty \mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/2\mathbb{Z})$ summand of $\mathcal{G}$. In this chapter, by focusing on order four algebraic concordance classes, we will see that

$$\mathcal{G} \cong (\oplus_1^\infty \mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/2\mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/4\mathbb{Z}).$$

We will also discuss infinite order and order 2 classes in more detail. Our goal is to obtain computable invariants that help determine these orders. As in the previous chapter, some of the discussion here is purely algebraic. We follow the treatment of [?]. See [?] for an alternative number theoretic approach to the classification of $\mathcal{G}$, presented by Stoltzfus.

Recall the real and p-adic completions of $\mathbb{Q}$ discussed in Section 5.4, and note the following fact.

Theorem 6.0.1. An isometric structure is trivial in $\mathcal{G}_{\mathbb{Q}}$ if and only if its extensions are trivial in $\mathcal{G}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{Q}_p}$, for each prime p.

See [?, Proposition 17]; compare the Hasse-Minkowski theorem in [?, 66:4] for quadratic forms.

6.1. Discriminants, resultants and decompositions of modules

Let's begin by discussing two invariants that we will soon need, namely, the discriminant Disc of a polynomial as well as the resultant Res of two polynomials. (Note that disc (with lower case "d") refers to the discriminant of a form, which is defined in 5.1.2.) More details regarding discriminants and resultants of polynomials can be found in basic algebra texts, for instance [?].

DEFINITION 6.1.1. For a monic polynomial $p = t^n + \cdots + a_0 \in \mathbb{F}[t]$ the discriminant is defined to be

$$\mathrm{Disc}(p) = \prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)^2,$$

where the α_i form a complete set of roots of p in the algebraic closure of $\mathbb{F}$. (Thus, if p has multiple roots, $\mathrm{Disc}(p) = 0$.)

DEFINITION 6.1.2. Given a second monic polynomial $q(t) = t^m + \cdots + b_0 \in \mathbb{F}[t]$, the resultant of the polynomials is defined to be

$$\mathrm{Res}(p, q) = \prod_{1 \le i \le n, 1 \le j \le m} (\alpha_i - \beta_j),$$

where the β_j are a complete set of roots of q.

The discriminant and resultant have explicit descriptions as (integer) polynomials in the coefficients of p and q. Thus, if $p, q \in R[t]$ where R is an principal ideal domain with field of fractions $\mathbb{F}$, then $\mathrm{Disc}(p) \in R$ and $\mathrm{Res}(p, q) \in R$. This can also be seen by noting that each is fixed by the appropriate Galois group, so it is in $\mathbb{F}$, and is an algebraic integer, so it is in R.

THEOREM 6.1.3. If $p, q \in R[t]$ are distinct irreducible polynomials, then there are polynomials $a, b \in R[t]$ such that $ap + bq = \mathrm{Res}(p, q)$.

As a corollary, there is the following result.

COROLLARY 6.1.4. Suppose that T is an automorphism of $V = R^n$ with characteristic polynomial Δ_T having irreducible factorization $\Delta_T = \prod g_i^{\epsilon_i}$. If $Res(g_i, g_j)$ is a unit for all $i \ne j$, then $V = \oplus V^{g_i}$, where V^{g_i} is invariant under T and T restricted to V^{g_i} has characteristic polynomial $g_i^{\epsilon_i}$.

To apply this result, it is easier to work with a single discriminant rather than all the resultants. In fact, we will be working with polynomials $\Delta \in \mathbb{Q}[t]$ and considering perhaps unknown factorizations in $\mathbb{Q}_p[t]$.

LEMMA 6.1.5. If p, q are distinct irreducible monic polynomials in $R[t]$ then $\mathrm{Res}(p, q)$ divides $\mathrm{Disc}(pq)$. In particular, if $\mathrm{Disc}(pq)$ is a unit in R, then so is $\mathrm{Res}(p, q)$.

Recall that $\Delta_A(t) = \det(A - tA^T)$ is the Alexander polynomial. Let $\bar{\Delta}_A(t)$ denote the product of the irreducible factors of $\Delta_A(t)$, and Disc denotes the discriminant of a polynomial, reviewed above. Returning to Theorem 6.0.1, our first result shows that while considering a specific example, we need not worry about all possible p–adic extensions, but instead restrict to a finite set of primes.

THEOREM 6.1.6. For a nonsingular integral Seifert matrix A, the class $[A] \in \mathcal{G}^{\mathbb{Z}}$ is of infinite order if and only if it is nontrivial in $\mathcal{G}_{\mathbb{R}}$; if it is of finite order, it is of order 4 if and only if it is of order 4 in $\mathcal{G}_{\mathbb{Q}_p}$ for some p dividing $\Delta_A(-1)$ with $p \equiv 3 \mod 4$; if it is of order 2, then it is of order 2 in $\mathcal{G}_{\mathbb{Q}_p}$ for some prime p dividing $2\det(A)\,\mathrm{Disc}(\bar{\Delta}_A(t))$

6.2. Elements of infinite order

As seen in 5.2.4, 5.2.5 and 5.2.6,Witt groups $W(\mathbb{Q}_p)$ are all finite. It follows that any element of infinite order in $\mathcal{G}_{\mathbb{Q}}$ is of infinite order in $\mathcal{G}_{\mathbb{R}}$. By Theorem 4.9.1, a class $(V, B, S) \in \mathcal{G}_{\mathbb{R}}$ splits as the direct sum of classes in $\mathcal{G}^{\delta}_{\mathbb{R}}$ where δ is an irreducible symmetric real polynomial. The only such polynomials are, up to a unit, of the form $t^2 + 2at + 1$, where $a^2 < 1$. The complex roots of this polynomial are the unit complex numbers ω, where $\omega = e^{i\theta}$ and $\cos\theta = a$.

THEOREM 6.2.1. The surjective signature function $\sigma\colon \mathcal{G}^{\delta}_{\mathbb{R}} \to 2\mathbb{Z} \subset \mathbb{Z}$ defined by $\sigma(V, B, S) = \sigma(B)$ is an isomorphism.

Proof As seen in 5.2.3, the signature function defines an isomorphism of $W(\mathbb{R})$ with $\mathbb{Z}$. Thus, by Theorem 4.9.9, a nontrivial form $(V, B, S) \in \mathcal{G}^{\delta}_{\mathbb{R}}$ in the kernel of the signature function on $\mathcal{G}^{\delta}_{\mathbb{R}}$ would have signature 0 and $\Delta_S(t) = \delta(t)^k$ for some odd $k = 2m+1$. This can be seen to be impossible as follows. If k is odd, then V is of dimension $4m+2$. The determinant of B is given, modulo squares, by $\delta(1)\delta(-1) = (2+2a)(2-2a) = 4(1-a^2) > 0$. However, a diagonal form over reals of dimension $4m + 2$ of signature 0 has determinant -1. □

The following theorem provides the means of computing the associated signatures.

THEOREM 6.2.2. The signature function $\mathrm{sign}((1-\omega)A+(1-\bar{\omega})A^T)$ defined for $\omega \in S^1$ has jumps only at the unit roots of the Alexander polynomial. If $\omega = e^{i\theta}$, with $\cos\theta = a$, is a root of $\Delta_A(t)$, then $\delta_a(t) = t^2 + 2at + 1$ is a factor of $\Delta_A(t)$ and the jump in the signature $\mathrm{sign}((1 - \omega)A + (1 - \bar{\omega})A^T)$ at ω equals, up to sign, the signature of $A + A^T$ restricted to $\mathcal{G}^{\delta_a}_{\mathbb{R}}$.

Proof See Exercise 2.10.14. □

6.3. Classes of order 4

In this section we show that all classes of order 4 in $\mathcal{G}_{\mathbb{Q}}$ in the image of $\mathcal{G}^{\mathbb{Z}}$ remain of order 4 when projected to $\mathcal{G}_{\mathbb{Q}_p}$ for some $p|\Delta_A(-1)$, $p \equiv 3 \mod 4$, where A is an integer matrix representing the class in $\mathcal{G}^{\mathbb{Z}}$. We also develop simple effective criteria to detect elements of order 4 that do not require a detailed p–adic analysis. The results here strengthen a result of Morita [**?**] in which it was shown that one can restrict to primes p dividing $2\Delta_A(1)\Delta_A(-1)$.

The restriction to $p \equiv 3 \mod 4$ is automatic, given that $W(\mathbb{F}_p)$ does not contain 4–torsion if $p \equiv 1 \mod 4$. The hardest technical work is in ruling out $p = 2$.

In general, Theorem 6.1.6, calling on an analysis of Witt groups over the p–adics, can be difficult to apply. A special case of a theorem of Levine (Section 23 of [**?**]) gives the following result.

THEOREM 6.3.1. Suppose that $\Delta_A(t)$ is an irreducible quadratic. Then A is of finite order in the algebraic concordance group if and only if $\Delta_A(1)\Delta_A(-1) < 0$. In this case A is of order 4 if $|\Delta_A(-1)| = p^a q$ for some prime p congruent to 3 modulo 4, a odd, and p and q relatively prime; otherwise it is of order 2.

EXAMPLE 6.3.2. We discussed the following Seifert form for the pretzel knot $K(1,-5,1)$ in Example 5.4.10 of Section 5.4 and showed that it maps to an order 4 element of $W(\mathbb{F}_3)$ implying that its order is ∞ or a multiple of 4. The above theorem shows that it is, in fact, order 4.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -5 \end{pmatrix} \implies A - tA^T = \begin{pmatrix} 1-t & 1 \\ -t & -5(1-t) \end{pmatrix}.$$

The determinant of which yields

$$\Delta_A(t) = -5t^2 + 11t - 5.$$

Clearly, $\Delta_A(1)\Delta_A(-1) = -3 \times 7$ which is negative, and either 3 or 7 provides the required $p \equiv 3 \bmod 4$ in Theorem 6.3.1 showing order 4.

THEOREM 6.3.3. If A is an integral Seifert matrix representing a class of order 4 in $\mathcal{G}^{\mathbb{Z}}$, then for some $p \equiv 3 \mod 4$ and some symmetric irreducible factor g of $\Delta_A(t)$, p divides $g(-1)$ and g has odd exponent in $\Delta_A(t)$.

THEOREM 6.3.4. If a class $\alpha \in \mathcal{G}_{\mathbb{Q}}$ that arises from a knot K, or, equivalently, in the image of $\mathcal{G}^{\mathbb{Z}}$, is of order 4, then α is of order 4 in $\mathcal{G}_{\mathbb{Q}_p}$ for some $p \equiv 3 \mod 4$ with p dividing $\Delta_A(-1)$.

The proof will use the following lemma.

LEMMA 6.3.5. Let $(V, B, S) \in \mathcal{G}^{\delta}_{\mathbb{Q}_p}$ be an isometric structure, where p is odd and $\delta \in \mathbb{Q}_p[t^{\pm 1}]$ is monic, irreducible and symmetric. If $\Delta_S(1)\Delta_S(-1) = p^{2e}u$ where u is a unit in $\mathbb{Z}_p$, then (V, B, S) is not of order 4. In particular, since $\Delta_S(t) = \delta(t)^k$, if $\delta(1)\delta(-1) = p^{2e}u$, then (V, B, S) is not of order 4.

Proof [Proof of Lemma 6.3.5] The form B can be diagonalized to be $[d_1, d_2, \ldots, d_k, pd_{k+1}, \ldots, pd_{2d}]$, where the d_i are units in $\mathbb{Z}_p$. In $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$, the determinant of the form is $\Delta_S(1)\Delta_S(-1) = p^{2e}u$. Thus k is even, say $k = 2l$.

Under the isomorphism of $W(\mathbb{Q}_p) \cong W(\mathbb{F}_p) \oplus W(\mathbb{F}_p)$, B maps to the pair of forms $[d_1, d_2, \ldots, d_{2l}] \oplus [d_{2l+1}, \ldots, d_{2d}]$. But a form of order 4 in $W(\mathbb{F}_p)$ is of odd rank. Thus, B is of order at most 2, and, applying Theorem 4.9.9, $2(V, B, S)$ is Witt trivial. $\square$

Proof [Proof Theorem 6.3.4] Let (V, B, S) be the rational isometric structure that arises from the Seifert matrix A representing a class in $\mathcal{G}^{\mathbb{Z}}$.

Fix for now a prime number p that does not divide $\Delta_A(-1)$. We will show that (V, B, S) cannot represent an element of order 4 in $\mathcal{G}_{\mathbb{Q}_p}$.

Recall that $\Delta_A(t) = \det(A)\Delta_S(t)$. By Gauss's Lemma, applied in the setting $\mathbb{Z}_p \subset \mathbb{Q}_p$, we can form the p–adic irreducible factorization $\Delta_A = \prod_i \dot{\delta}_i \prod_j \dot{f}_j$ where the $\dot{\delta}_i \in \mathbb{Z}_p[t]$ are the symmetric factors and the remaining factors, the $\dot{f}_j \in \mathbb{Z}_p[t, t^{-1}]$, occur in $(t \to t^{-1})$ conjugate pairs. The dots over the polynomials indicate that these are associates (differ by multiplication by a nonzero element of $\mathbb{Q}_p$) of the irreducible monic factors of Δ_S.

If (V, B, S) is of order 4 in $\mathcal{G}_{\mathbb{Q}_p}$, then the image of (V, B, S) in $\mathcal{G}^{\delta_i}_{\mathbb{Q}_p}$ will be of order 4 for one of the δ_i, which we now denote δ. Call this image $(V_\delta, B_\delta, S_\delta)$.

Case I, p **odd:** (Morita's theorem) Since p does not divide $\Delta_A(1)\Delta_A(-1)$, this product is a unit in $\mathbb{Z}_p$, and the same is true for $\dot{\delta}(1)\dot{\delta}(-1)$. It follows that $\delta(1)\delta(-1)$ is of the form a^2u where u is a unit, and thus Lemma 6.3.5 applies to show that $(V_\delta, B_\delta, S_\delta)$ is not of order 4.

Case II, $p = 2$: Recall that the *discriminant* of a form B over $\mathbb{F}$ of even rank $2e$ is defined to be $\mathrm{disc}(B) = (-1)^e \det(B)$. This determines a homomorphism $I \to \mathbb{F}^*/(\mathbb{F}^*)^2$, where I is the subgroup of $W(\mathbb{F})$ generated by forms of even rank.

In the present situation, as mentioned above, we have from [?] that $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$ is of order 8, with representatives $\{\pm 1, \pm 2, \pm 5, \pm 10\}$. One then checks immediately that the values of the discriminants of the classes of order 4 are $\{-1, -2, -5, -10\} \subset \mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$. It remains to show that $\mathrm{disc}(B_\delta)$ is not in $\{-1, -2, -5, -10\}$ modulo squares.

The characteristic polynomial of S_δ is, up to a constant, $g(t)$ where $g(t)$ is a symmetric polynomial with coefficients in $\mathbb{Z}_2$. Since $\Delta_A(1) = 1$ and $g(t)$ is a factor of $\Delta_A(t)$ in the $\mathbb{Z}_p$–adic factorization, $g(1)$ is a unit in $\mathbb{Z}_2$, and so after multiplying by another constant we can assume that $g(1) = 1$. Given that g is symmetric and of even degree $2e$, we write

$$g(t) = a_0 + a_1 t + \cdots + a_{e-1}t^{e-1} + a_e t^e + a_{e-1}t^{e+1} + \cdots + a_0 t^{2e}.$$

Since $g(1) = 1$, we have $a_e = 1 - 2a_{even} - 2a_{odd}$, where a_{even} and a_{odd} are the sums of the coefficients with even or odd index, respectively.

Since the determinant of B_δ is given by $g(1)g(-1)$ modulo squares, the discrimant of B_δ is given by $(-1)^e g(1)g(-1) = (-1)^e g(-1)$, which expanded equals

$$(-1)^e(2a_{even} - 2a_{odd} + (-1)^e(1 - 2a_{even} - 2a_{odd}) = 1 + 4a^*,$$

for some a^*. Thus, $\mathrm{disc}(B) \equiv 1 \mod 4$. None of the elements in $\{-1, -2, -5, -10\}$ are equivalent to $1 \mod 4$ and thus B_δ is not of order 4 in $W(\mathbb{Q}_2)$. $\square$

The following Corollary is an obvious consequence.

Corollary 6.3.6. *If $\Delta_A(-1)$ has no prime factor p with $p \equiv 3 \mod 4$, then K is not of order 4 in $\mathcal{G}$.*

Corollary 6.3.7. *If A is of order 4 in $\mathcal{G}$ then for some $p \equiv 3 \mod 4$ and some symmetric irreducible factor $g(t) \in \mathbb{Z}[t, t^{-1}]$ of $\Delta_A(t)$, p divides $g(-1)$ and g has odd exponent in Δ_A.*

Proof [Proof of Corollary 6.3.7] If A has order 4, this will be detected in $\mathcal{G}_{\mathbb{Q}_p}$ for some $p \equiv 3 \mod 4$ that divides $\Delta_A(-1)$. In turn, it will be detected in $\mathcal{G}^\delta_{\mathbb{Q}_p}$ for some δ that divides $\Delta_A(t)$. Suppose that $g(t)$ is the irreducible factor of $\Delta_T(t)$ that is divisible by $\delta(t)$. If $g(t)$ has even exponent in $\Delta_T(t)$, then Δ_T^δ will be an even power of δ. Thus, according to Lemma 6.3.5, $2(V^\delta, B^\delta, S^\delta) = 0 \in \mathcal{G}^\delta_{\mathbb{Q}_p}$.

$\square$

Although the previous results permit one to avoid working over the p–adics in many cases, we want to present one example that illustrates the necessity of undertaking a p–adic analysis.

Consider a knot K with Alexander polynomial $\Delta(t) = 4t^2 - 2t^3 - 3t^2 - 2t^3 + 4$. This polynomial is irreducible over $\mathbb{Q}$ and has no roots on the unit circle. Since $\Delta(1)\Delta(-1) = 9$, K is of order either 2 or 4. Furthermore, if K is of order 4, it will be detected at $p = 3$. In fact, $\Delta(t)$ factors as the product of two distinct symmetric irreducible quadratic polynomials $\delta_1(t)$, $\delta_2(t)$ over $\mathbb{Z}_3$, each of which satisfies $\delta_i(1)\delta_i(-1)$ is divisible by 3. Thus, K will be of order 4 in $\mathcal{G}$.

To find this factorization, consider a factorization as $\Delta(t) = (2+at+2t^2)(2+bt+2t^2)$. One quickly computes that $b = -1 - a$ and that $a^2 + a - 5 = 0$.

6.4. Classes of order 2

In this section we consider forms $(V, B, S) \in \mathcal{G}_{\mathbb{Q}}$ that are known to be of finite order and not of order 4. There are two cases, one of which is trivial.

6.4.1. The trivial case: odd exponent. Suppose the $\Delta_S(t)$ has a symmetric irreducible factor with odd exponent. Then (V, B, S) is nontrivial, and so of order exactly 2.

6.4.2. The even exponent case. We are reduced to the case that (V, B, S) is of order 1 or 2, and all irreducible symmetric factors of $\Delta_S(t)$ have even exponent. In the case of identifying order 4 classes, we saw that primes that divide the determinant of the class were key. Here we must also consider the discriminant of the polynomial, $\mathrm{Disc}(\Delta_S)$, and $\det(B)$. The definition of these is presented in 6.1.

Theorem 6.4.1. *Let A be a nonsingular Seifert matrix representing a class in $\mathcal{G}^{\mathbb{Z}}$ of rank $2g$ and let $(\mathbb{Q}^{2g}, A + A^T, A^{-1}A^T) = (V, B, S) \in \mathcal{G}_{\mathbb{Q}}$. Suppose that all irreducible symmetric factors of $\Delta_A(t)$ have even exponent. Then for any prime p that does not divide $2\det(A)\,\mathrm{Disc}(\bar{\Delta}_A(t))$, $(V, B, S) = 0 \in \mathcal{G}_{\mathbb{Q}_p}$, where $\bar{\Delta}_A(t)$ denotes the product of all the distinct irreducible factors of Δ_A.*

Proof We begin by defining $\mathcal{G}_{\mathbb{Z}_p}$. This is the Witt group consisting of triples (V, B, S) where V is a free $\mathbb{Z}_p$–module, B is a symmetric bilinear form on V with determinant a unit in $\mathbb{Z}_p$, and S is an isometry of (V, B).

We show in Lemma 6.4.2 below that since p does not divide $\mathrm{Disc}(\bar{\Delta}_V)$, $det(A + A^T)$ is a unit in $\mathbb{Z}_p$. Furthermore, the entries of $A^{-1}A^T$ are rational with denominators prime to p and so all entries are elements of $\mathbb{Z}_p$. It follows that this matrix defines an isometry of $\mathbb{Z}_p^{2g}$. Thus, the class $(\mathbb{Q}^{2g}, A + A^T, A^{-1}A^T) \in \mathcal{G}_{\mathbb{Q}}$ is in the image of the class $(\mathbb{Z}_p^{2g}, A + A^T, A^{-1}A^T) \in \mathcal{G}_{\mathbb{Z}_p}$

The characteristic polynomial Δ of $A^{-1}A^T$ is a monic polynomial in $\mathbb{Z}_p[t^{\pm 1}]$, and has irreducible factorization over $\mathbb{Z}_p$ (and $\mathbb{Q}_p$) as $\prod_i \delta_i^{\epsilon_i} \prod_j g_j$, where the δ_i are the irreducible symmetric factors and the ϵ_i are all assumed to be even.

Since p is prime to the discriminant,$\mathrm{Disc}(\Delta)$, as we describe in 6.1 the splitting $(V, B, S) = \oplus(V_{\delta_i}, B_{\delta_i}, S_{\delta_i})$ can be viewed as a splitting in $\mathcal{G}_{\mathbb{Z}_p}$, rather than in $\mathcal{G}_{\mathbb{Q}_p}$.

It remains to show that each one of these summands, say $(V_\delta, B_\delta, S_\delta)$, is Witt trivial as a class in $\mathcal{G}^{\delta}_{\mathbb{Q}_p}$. Since the exponent is even, by Theorem 4.9.9 it is sufficient to show that $(V^\delta, B^\delta) = 0 \in W(\mathbb{Q}_p)$. Notice that since the exponent on δ is even, V has rank $4m$ for some m and the determinant (and thus the discriminant) of B is a square, and so is trivial in $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$.

Diagonalize B^δ to be $[u_1, \ldots, u_k, pv_1, \ldots, pv_j]$. There is the homomorphism

$$\phi^o \colon W(\mathbb{Q}_p) \to W(\mathbb{F}_p)$$

that takes our diagonalized class to $[v_1, \ldots, v_j]$. However, as described in 5.1, ϕ^o vanishes on $W(\mathbb{Z}_p)$. Thus, $[v_1, \ldots, v_j]$ is of even rank and has discriminant 1. It follows that applying ϕ^e to our form (resulting in $[u_1, \ldots, u_k]$) is also a form of even rank and discriminant 1, and so is trivial in $W(\mathbb{F}_p)$.

Since p is odd, $\phi^o \oplus \phi^e$ defines an isomorphism $W(\mathbb{Q}_p) \to W(\mathbb{F}_p) \oplus W(\mathbb{F}_p)$. Thus we see that (V^δ, B^δ) is Witt trivial.

$\square$

LEMMA 6.4.2. *Let A be a Seifert matrix and suppose the prime p does not divide* $\det(A)\,\mathrm{Disc}(\bar{\Delta}_A(t))$. *Then* $\det(A + A^T)$ *is a unit in* $\mathbb{Z}_p$.

Proof We begin by noting that $\det(A + A^T) = \Delta_A(1)\Delta_A(-1)$. This will be a unit if and only if $\bar{\Delta}_A(1)\bar{\Delta}_A(-1)$ is a unit; removing multiple factors does not change whether an element in $\mathbb{Z}$ is divisible by p. The leading coefficient of $\bar{\Delta}_A(t)$ is a divisor of $\det(A)$, and so is prime to p and is a unit in $\mathbb{Z}_p$. Dividing by that leading coefficient yields a monic polynomial $\bar{\Delta}_S(t) \in \mathbb{Z}_p$. We need to show that $\bar{\Delta}_S(1)\bar{\Delta}_S(-1)$ is a unit in $\mathbb{Z}_p$.

The discriminant of $\bar{\Delta}_S(t)$ is given by $\mathrm{Disc}(\bar{\Delta}_S(t)) = \prod_{i,j}(\alpha_i - \alpha_j)^2$ where the product is over all distinct pairs of roots of $\bar{\Delta}_S(t)$ in the algebraic closure of $\mathbb{Q}_p$. Since $\bar{\Delta}_S(t)$ is symmetric and does not have ± 1 as a root, if α is a root, then so is $\frac{1}{\alpha} \neq \alpha$. Collecting roots that occur in inverse pairs, we find

$$\mathrm{Disc}(\bar{\Delta}_S(t)) = \prod(\alpha_i - \frac{1}{\alpha_i})^2 \prod(\alpha_i - \alpha_j)^2,$$

where the second product is taken over pairs with $\alpha_j \neq \frac{1}{\alpha_i}$.

This product can be rewritten as

$$\frac{1}{\prod \alpha_i^2} \prod_i (\alpha_i^2 - 1)^2 \prod(\alpha_i - \alpha_j)^2 = \frac{1}{\prod_i \alpha_i^2} \prod(\alpha_i - 1)^2(\alpha_i + 1)^2 \prod(\alpha_i - \alpha_j)^2.$$

Since $\bar{\Delta}_S(t)$ is monic and symmetric, $\prod \alpha_i = 1$ and the entire product can be simplified to give

$$\mathrm{Disc}(\bar{\Delta}_S(t)) = \bar{\Delta}_S(1)^2 \bar{\Delta}_S(-1)^2 \prod(\alpha_i - \alpha_j)^2.$$

The two elements $\bar{\Delta}_S(1)$ and $\bar{\Delta}_S(-1)$ are clearly in $\mathbb{Z}_p$, and we are assuming that $\mathrm{Disc}(\bar{\Delta}_S(t))$ is a unit in $\mathbb{Z}_p$. Thus, if we show that $\prod(\alpha_i - \alpha_j)^2$ is in $\mathbb{Z}_p$, each of $\bar{\Delta}_S(1)$, $\bar{\Delta}_S(-1)$, and $\prod(\alpha_i - \alpha_j)^2$ are seen to be units in $\mathbb{Z}_p$.

To see that $\prod(\alpha_i - \alpha_j)^2$ is in $\mathbb{Z}_p$, note that it is fixed by the Galois group of the splitting field of $\bar{\Delta}_S(t)$ over $\mathbb{Q}_p$, and thus is in $\mathbb{Q}_p$. However, it is an algebraic integer in the algebraic closure of the fraction field of $\mathbb{Z}_p$. The only algebraic integers in the fraction field of an integral domain must be in the domain itself. Thus, it is in $\mathbb{Z}_p$ as desired. □

6.5. The algebraic order of prime knots with 12 or fewer crossings

There are 2,977 prime knots with 12 or fewer crossings. Here we describe how the algebraic orders of all such knots are determined. Our goal is to present enough of the calculation to illustrate how the complete set of results, appearing in the *KnotInfo* table [?], were derived. In addition, we have isolated out special cases to demonstrate methods that readily apply when specific theorems cannot be quoted directly.

Infinite order: If a knot has infinite order, it is detected by a nontrivial signature. For 2,132 of the knots, the signature of $A + A^T$ is nonzero. Another 125 knots have nontrivial

ω–signature (the signature of $(1-\omega)A + (1-\omega^{-1})A^T$) nontrivial for some unit complex number ω. This leaves 720 knots of finite algebraic order.

Slice knots: There are 157 knots that have been identified as topologically slice (many through the unpublished work of Stoimenow [**?**]). This leaves 563 knots to resolve.

Order 4: By Theorem 5.4.9, if K is of finite algebraic order and $D = \Delta_K(-1) = \det(A + A^T)$ is divisible by a prime p, $p \equiv 3 \mod 4$ and p has odd exponent in D, then K is of order 4. This applies to 172 of the remaining knots, leaving 391 to resolve.

Order 2: If K is of finite algebraic order and its Alexander polynomial has a symmetric factor (over $\mathbb{Q}$) with odd exponent, it has order 2 or 4. If no prime $p \equiv 3 \mod 4$ divides $\Delta_K(-1)$ then K is of order 2 in $\mathcal{G}$. This applies to 318 of the remaining knots, leaving 73 to resolve.

More order 2: As in the previous paragraph, if K has finite algebraic order and its Alexander polynomial has a symmetric factor (over $\mathbb{Q}$) with odd exponent, it has order 2 or 4. If in addition K is amphicheiral (equal to its mirror image, regardless of orientation) it is of algebraic order exactly 2. (Reversing the orientation of a knot has the effect of transposing the Seifert matrix A. An examination of Levine's classification reveals the this does not change the algebraic concordance class of a knot. As a more explicit proof, see for instance [**?**].) This applies to another 5 knots, leaving 68 to resolve.

Basic examples of algebraically slice knots: If the Alexander polynomial has no irreducible symmetric factors, the knot is algebraically slice. This applies to another 9 knots, leaving 59 cases to resolve.

More order 2: Of these remaining 59 knots, 25 have the property that an irreducible factor $g(t)$ of the Alexander polynomial has odd exponent, so that the knot is of order 2 or 4, but no such irreducible factor has $g(-1)$ divisible by a prime $p \equiv 3 \mod 4$. Thus, by Corollary 6.3.7, the knot has order exactly 2. As an example, the knot 9_{24} has Alexander polynomial $1 - 5t + 10t^2 - 13t^3 + 10t^4 - 5t^5 + t^6$ with determinant $45 = 3^2 5$. Thus it is conceivable that it could be of order 4, detected at the prime 3. But the polynomial factors as $(1 - 3t + t^2)(1 - t + t^2)^2$, and the 3 factor arises from a symmetric irreducible factor of exponent 2. As a second example, the knot 9_{34} has Alexander polynomial with determinant $3^2 5$. In this case the polynomial factors as $(-2 + t)(-1 + 2t)(1 - 3t + t^2)$ and thus the 3 factor does not arise from a symmetric factor of the Alexander polynomial. All 25 cases are similar to one of these two examples.

This leaves 34 knots to consider.

More algebraically slice: There are nine remaining knots for which all symmetric factors have even degree. These are either trivial in $\mathcal{G}_{\mathbb{Q}}$ or are of order 2. We can rule out order 2 in all the cases to see that these are algebraically slice, as follows.

For seven of these knots, there is a unique symmetric irreducible factor δ, and it is of degree 2. For instance, for 12_{a169} the Alexander polynomial factors as $(2-3t+2t^2)^2$ and for 12_{n224} the Alexander polynomial factors as $(1-2t)(2-t)(1-t+t^2)^2$. In this case, regardless of the prime p, if the quadratic factors over $\mathbb{Q}_p$, then the form is Witt trivial, since no even degree symmetric factors would remain. If the quadratic is irreducible, then the form $B = A + A^T$ would be Witt trivial if and only if the form B_δ is Witt trivial, since the form B restricted to the complement of the δ summand is automatically Witt trivial. In each case, one can diagonalize the form $V + V^t$ and find that the diagonal elements are paired $[d_1, -d_1, d_2, -d_2, \ldots]$.

One of the two more challenging cases is that of $K = 12_{a990}$. There is a nonsingular Seifert matrix A for K of size 8×8, and $\Delta_K(t) = (t^2-t+1)^2(t^2-3t+1)^2$. Thus the corresponding transformation S has characteristic polynomial $(t^2+t+1)^2(t^2+3t+1)^2$. Letting $\delta_1(t) = t^2+t+1$ and $\delta_2(t) = t^2+3t+1$ it is easy to find a basis for V_{δ_1} and V_{δ_2} over the rationals. These are just the images of the transformations $\delta_2(S)^2$ and $\delta_1(S)^2$ respectively, both of which are rank 4. On each of these, it is easy to find the respective quadratic form, simply by restricting B to each, and these can be diagonalized over the rationals, with diagonal entries integers. For some primes p, δ_i might factor over $\mathbb{Q}_p$, but since it is quadratic, if it factors then the form is automatically Witt trivial. So we assume that δ_i is irreducible over $\mathbb{Q}_p$. In this case, it is sufficient to show that the forms B_{δ_i} are trivial over the rationals (the exponent of the characteristic polynomial is even). For this, one can apply the classification of Witt forms over $\mathbb{Q}$, as described in Section 5.1. This calls for a consideration of all prime integers, but if p does not divide any of the diagonal entries of B_{δ_i}, then one notes that the induced forms in $W(\mathbb{F}_p)$ are of rank 4 with trivial discriminant (that is, a square), and thus the forms are Witt trivial. At the finite set of primes that remain, one must check that the image forms in $W(\mathbb{F}_p)$ are trivial. In the actual calculation for these examples, only four primes appear (though this might depend on the choice of spanning set of V_{δ_i}) and so the calculation is quickly done.

The second case that requires further calculation is that of 12_{n681} which has a nonsingular Seifert matrix of size 8×8 and Alexander polynomial $(t^4 - t^3 + t^2 - t + 1)^2$, so that the corresponding transformation S has characteristic polynomial $(t^4 - t^3 + t^2 - t + 1)^2$. Letting $\delta(t) = t^4 - t^3 + t^2 - t + 1$, one finds the image of the transformation $\delta(S)$ is a rank 4 invariant subspace of the 8–dimensional rational vector space on which the form vanishes.

Generalizing this example

In the case that the Alexander polynomial of K factors as $\delta(t)^2$ with $\delta(t)$ irreducible, if the transformation $\delta(S)$ is nontrivial then K is algebraically slice. The proof consists of noting that the $\mathbb{Q}[t^{\pm 1}]$–module V is of the form $\mathbb{Q}[t^{\pm 1}]/ < \delta(t)^2 >$. The image of $\delta(S)$ will be half dimensional and the form B vanishes on the image. In the previous example, that of K_{a990}, this approach does not work: as a $\mathbb{Q}[t^{\pm 1}]$–module V is isomorphic to $\mathbb{Q}[t^{\pm 1}]/ < (t^2 + 3t + 1)^2 > \oplus \mathbb{Q}[t^{\pm 1}]/ < t^2 + t + 1 > \oplus \mathbb{Q}[t^{\pm 1}]/ < t^2 + t + 1 >$ and thus there are an infinite number of invariant submodules to consider.

Order 2: The remaining 25 cases are the most technical. In these cases there is a unique prime $p \equiv 3 \mod 4$ dividing $\Delta_K(-1)$ and in each case p has exponent 2 in $\Delta_K(-1)$. Furthermore, there is an irreducible factor f of Δ_K (over $\mathbb{Z}$) such that f has exponent 1 in Δ_K and p^2 divides $f(-1)$. If f remains irreducible in $\mathbb{Q}_p$ then arguments as given above would imply that K is of order exactly 2.

However, it is conceivable that K will be of order 4, but for this to occur, f would factor as $f = f_1 f_2 \cdots f_n$ over $\mathbb{Q}_p$ (and thus over $\mathbb{Z}_p$), and for at least one of the symmetric f_i (and so for at least two of the symmetric f_i), $f_i(1)f_i(-1) \neq 1 \in \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$, or, put otherwise, $D(f_i(1)f_i(-1)) \neq 1 \in \mathbb{Z}/2\mathbb{Z}$ (where $D\colon\ \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 \to \mathbb{Z}/2\mathbb{Z}$ is defined in Section 5.4). Call two of these factors f_a and f_b.

There is the canonical homomorphism $\mathbb{Z}_p \to \mathbb{Z}/p\mathbb{Z}$. Denote this map by $x \to \bar{x}$. Similarly, there is the induced map $\mathbb{Z}_p[t] \to (\mathbb{Z}/p\mathbb{Z})[t]$, which we denote by $f(x) \to \bar{f}(x)$.

The symmetric factors f_a and f_b are both of even degree, and thus $\bar{f}_a$ and $\bar{f}_b$ are even degree and symmetric, after factoring out a power of t so that they have nonzero constant term.

Since $D(f_a(1)f_a(-1)) \neq 1$ (that is, $f(1)f(-1) = p^k u$ where k is odd and u is a unit), it must be the case that $\bar{f}_a(1)\bar{f}_a(-1) = 0 \in \mathbb{Z}/p\mathbb{Z}$. Thus, $\bar{f}_a$ must be divisible by $t \pm 1$. Similarly for $\bar{f}_b$. We can show that each of the 25 knots in this category don't satisfy this criteria. A few examples follow.

The knot 11_{a300} has $\Delta_K(1)\Delta_K(-1) = 3^2 17$. Thus the only prime of interest is 3. When reduced modulo 3, we have the irreducible factorization $\bar{\Delta}_K(t) = (1+t)^2(1+t^2)(1+t+t^2+t^3+t^4)$. These factors cannot be combined to find two symmetric factors, each of which is of even degree and divisible by $t \pm 1$.

A similar example is 12_{a1170}, again with $\Delta_K(1)\Delta_K(-1) = 3^2 17$. Its Alexander polynomial reduced modulo 3 satisfies $\bar{\Delta}_K(t) = 2(1+t)^2(2+t+t^2+t^3)(2+2t+2t^2+t^3)$. In this case, by distributing the $1+t$ factors between the two other factors we split the polynomial into even degree polynomials, each of which evaluates trivially at $t = -1$. However, neither of these is symmetric.

This approach works for 24 of the 25 knots of this variety. The one exception is 12_{n525}. It has $\Delta_K(t) = 1 - 8t + 28t^2 - 43t^3 + 28t^4 - 8t^5 + t^6$. Again, $\Delta_K(1)\Delta_K(-1) = 3^2 17$.

Working modulo 3, this polynomial factors as $(1+t)^4(1+t^2)$. Suppose that $\Delta_K(t)$ factors nontrivially with two or more symmetric factors, each of even degree, over the 3–adics. One possibility would be that there are degree 2 and degree 4 irreducible factors. In this case, one possibility for the corresponding factorization modulo 3 would be $[(1+t^2)][(1+t)^4]$ and the other would be $[(1+t)^2][(1+t)^2(1+t^2)]$. The other possibility is that $\Delta_K(t)$ factors over the 3–adics as the product of three symmetric quadratics. Then, modulo 3, the corresponding factorization would be $[(1+t)^2][(1+t)^2][(1+t^2)]$.

Notice that among these three cases, there are only two cases in which there are two irreducible factors over the 3–adics both of which satisfy $\delta(1)\delta(-1) = 0 \mod 3$. In each of these two cases there is a quadratic factor which satisfies $\delta(1)\delta(-1) = 0 \mod 3$. We want to show that this does not occur.

One way to do this is to find the p–adic factorization. Another way is to check for factorizations modulo 3^k for various k. The second method can be done quickly by computer, and we find that modulo 27 the only factorization into a quadratic and quartic has quadratic term $1 + t^2 \mod 3$, and this does not satisfy $\delta(1)\delta(-1) = 0 \mod 3$.

We now want to describe a method for factoring over the p–adics. If Δ_K factored as desired, then we would have $\Delta_K = fg$, where $f(t) = 1 + at + t^2$ and $g(t) = 1 + bt + ct^2 + bt^3 + t^4$, where $a, b, c \in \mathbb{Z}_3$. (This uses Gauss's Lemma and the fact the $\Delta_K(t)$ is monic.)

For this to hold, we see immediately that $b = -8 - a$. Making this substitution into g and multiplying gives $fg - \Delta_K = (-27-8a-a^2+c)t^4 + (27-2a+ca)t^3 + (-27-8a-a^2+c)t^2$, so $c = 27 + 8a + a^2$.

Again substituting and expanding gives $fg - \Delta_K = (27+25a+8a^2+a^3)t^3$. The number $a = 0$ is a solution modulo 3 to the equation $h(a) = 27 + 25a + 8a^2 + a^3 = 0$. According to the general theory of p–adic polynomials, since 0 is not a solution to $h'(a) = 0$, it lifts to a p–adic solution. In this case, it is relatively easy to find that lifting: knowing the value mod 3 permits one to find the solution modulo 9; this solution then is easily lifted to a solution modulo 27, and so on. For instance, modulo 3^8 a solution is $a = 2565 = 2(3^3) + 1(3^4) + 1(3^5) + 1(3^7)$. The factorization of Δ_K modulo $3^8 = 656144$ is

$$\Delta_K(t) = (1 + 2565t + t^2)(1 + 3988t + 5967t^2 + 3988t^3 + t^4) \mod 3^8.$$

CHAPTER 7

Cyclic Covers and their homology

We have seen in Theorem 2.5.2 that the first homology group of the knot exterior $X(K)$ is the infinite cyclic group $\mathbb{Z}$. This allows us to form cyclic covering spaces of $X(K)$ with the group of deck transformations isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}/n$ for a positive integer n. Seifert form and Alexander polynomial are closely related to the homology of these covers. Before we discuss knot concordance in further depth, a review of cyclic covers of knot exteriors, branched cyclic overs of S^3, and brached covers of B^4 is in order as they provide important tools in this study.

7.1. The infinite cyclic cover

7.1.1. Covering space theory. Given a path connected CW-complex X, a group G, and a homomorphism $\phi\colon \pi_1(X) \to G$, there is an associated cover, defined as follows. Let $K(G,1)$ be an Eilenberg-McLane space, with universal cover $E(G)$, having covering map p and group of deck transformations identified with G. Let $\phi'\colon X \to K(G,1)$ be a map inducing ϕ on π_1. Then X_ϕ is defined to be

$$X_\phi\{(x,e) \in X \times E(G) \mid \phi'(x) = p(e)\}.$$

One can check that X_ϕ is a covering space of X via projection on the first factor. Also, each component of X_ϕ has fundamental group isomorphic to $\mathrm{Ker}(\phi)$. The action of G on the components of X_ϕ corresponds to the action of G on the cosets of $\mathrm{Im}(\phi)$.

7.1.2. Existence of infinite cyclic covers of knots. We begin with a summary of consequences of the general theory of covering spaces. Let $K \subset S^3$ be a knot, and suppose that $\chi\colon H_1(X(K)) \to \mathbb{Z}$ is a homomorphism. There is then an induced map $\chi_\pi\colon \pi_1(X(K)) \to \mathbb{Z}$. The kernel is a subgroup π_χ of π, with index $[\pi\colon \pi_\chi]$ equal to either 1 or infinite, depending on whether or not χ is trivial. There is a cover $\tilde{X}_\chi(K)$ of $X(K)$, with group of deck transformations isomorphic to $\mathbb{Z}$.

In the case that χ is trivial, $\tilde{X}_\chi(K)$ consists of an infinite collection of copies of $X(K)$, indexed by $\mathbb{Z}$, with deck transformation acting by shifting the indices. In the case that χ is surjective, the cover is connected.

If χ is onto an index k subgroup, there is an isomorphism of $\mathrm{Im}(\chi)$ with $\mathbb{Z}$. Let χ' denote the composition of χ with this isomorphism. Then there is the connected cover $\tilde{X}_{\chi'}(K)$. The cover $\tilde{X}_\chi(K)$ consists of k copies of $\tilde{X}_{\chi'}(K)$, indexed with the integers $1, \dots k$. Momentarily, call them $\tilde{X}_i$. The generating deck transformation acts by shifting $\tilde{X}_i$ to $\tilde{X}_{i+1}$ for $i < k$ and sending $\tilde{X}_k$ to $\tilde{X}_1$, via the deck transformation of $\tilde{X}_{\chi'}(K)$.

7.1.3. A geometric decomposition of the infinite cyclic cover. We now restrict to the case that χ is surjective and describe an explicit construction of $\tilde{X}_\chi(K)$. A more detailed description with examples can be found in [**?**]. Let F be a Seifert surface of K with the property that $\chi(\alpha) = \alpha \cap F$ for $\alpha \in H_1(X(K))$. (If $K \subset S^3$, this is automatic.) Let $N(F)$ be a product neighborhood of F, identified with $F \times [-1, 1]$. For $i \in \mathbb{Z}$ let W_i be a copy of $X(K) - F$ and let N_i be a copy of $N(F)$. Let W be the disjoint union of the W_i and let N be the disjoint union of the N_i.

We define a quotient space of the disjoint union of W and N as follows. Notice that N_i contains a subspace homeomorphic to $F \times (0, 1]$, denoted N_i^+, and there is a homeomorphic subspace of W_{i+1}. Identify these subspaces. Similarly, N_i contains a subspace homeomorphic to $F \times [-1, 0)$, denoted N_i^-, and there is a homeomorphic subspace of W_i. Again, identify these subspaces. Momentarily, call the resulting space $\tilde{X}$. It is not hard to show that $\tilde{X}$ is a covering space with group of deck transformations isomorphic to $\mathbb{Z}$. By uniqueness of covering spaces, this cover must be $\tilde{X}_\chi(K)$. A schematic picture of the construction is given in Figure 7.1.

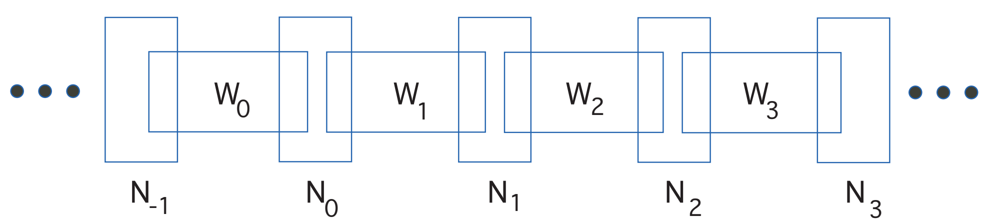

FIGURE 7.1. Schematic of Infinite Cyclic Cover

Alternative Construction There is a map $f\colon\ X(K) \to S^1$ inducing χ. Recall that S^1 is an Eilenberg-McLane space with universal cover $\mathbb{R}^1$. Thus, $\tilde{X}_\chi(K)$ can be defined as a subset of $X(K) \times \mathbb{R}^1 \subset S^3 \times \mathbb{R}^1$. One can now use these $\mathbb{R}^1$ coordinates to construct the geometric decomposition of the infinite cyclic cover given in the previous paragraph.

7.1.4. The homology of the infinite cyclic cover for $K \subset S^3$. The homology of $\tilde{X}_\infty(K)$ for knots in S^3 can now be described via the Mayer-Vietoris sequence:

$$0 \to H_2(\tilde{X}_\infty(K)) \to H_1(N^+) \oplus H_1(N^-) \to H_1(N) \oplus H_1(W) \stackrel{\text{inc}_1 - \text{inc}_2}{\to} H_1(\tilde{X}_\infty(K)) \to 0.$$

Notice that $H_1(N) \cong H_1(N_0) \otimes \mathbb{Z}[t, t^{-1}]$ as a $\mathbb{Z}[t, t^{-1}]$–module, where t acts on $H_1(N)$ via the deck transformation. Similarly isomorphisms exist for the other relevant homology groups. Treating all the groups as $\mathbb{Z}[t, t^{-1}]$–modules simplifies our calculations, and gives added information.

If a basis is chosen for $H_1(F)$, there are corresponding bases for $H_1(N_0)$, $H_1(W_0)$, $H_1(N_0^+)$, $H_1(N^-0)$, and, using a dual basis, $H_1(W_0)$. If A is the Seifert matrix for F, then the middle map is given by the block matrix:

$$\begin{pmatrix} I_{2g} & I_{2g} \\ tA^T & A \end{pmatrix}.$$

THEOREM 7.1.1. The first homology group of the infinite cyclic cover of a knot $K \subset S^3$ is presented by the matrix $A - tA^T$ as a $\mathbb{Z}[t, t^{-1}]$–module. Since $\det(A - tA^T)$ is nonzero ($\det(A - A^T) = \pm 1$), $H_2(\tilde{X}(K)) = 0$.

Rational Homology A finitely generated module M over a principal ideal domain (PID) P is isomorphic to a direct sum of cyclic modules:

$$M \cong \frac{P}{\langle p_1 \rangle} \oplus \cdots \oplus \frac{P}{\langle p_k \rangle}$$

where $\langle p_i \rangle$ denotes the principal ideal spanned by an element $p_i \in P$. The product ideal $\langle p_1 p_2 \cdots p_k \rangle$ is called the *order* of the module M. A generator of this module is the Alexander polynomial well-defined up to multiplication by units in P. See [**?**] for a detailed discussion.

In the above setting, switching coefficients to the rationals, $H_1(\tilde{X}(K); \mathbb{Q})$ is a finitely generated module over the ring $\mathbb{Q}[t, t^{-1}]$, and since $\det(A^T - tA)$ is nonzero, it is a torsion module. Since $\mathbb{Q}$ is a field, the ring $\mathbb{Q}[t, t^{-1}]$ is a principal ideal domain, so we have

$$H_1(\tilde{X}_\infty(K); \mathbb{Q}) \cong \mathbb{Q}[t, t^{-1}]/\langle f_1 \rangle \oplus \mathbb{Q}[t, t^{-1}]/\langle f_1 \rangle \oplus \cdots \oplus \mathbb{Q}[t, t^{-1}]/\langle f_n \rangle$$

for some polynomials f_i with $f_i | f_{i+1}$ for all $i < n$. The polynomials f_i, and thus their product are invariants of the module, and thus we define the Alexander polynomial to be the generator $\prod f_i(t)$ of the order ideal.

This is only defined up to multiples by units in $\mathbb{Q}[t, t^{-1}]$, which are of the form $at^i, a \in \mathbb{Q}$. Thus, there is always a representative which is an integer polynomial for which the greatest common divisor of the coefficients is 1. Since the determinant of $A - tA^T$ evaluates to be ± 1 at $t = 1$, this determinant satisfies that property. Thus, the Alexander polynomial is defined to be $\det(A - tA^T) \in \mathbb{Z}[t, t^{-1}]$. Compare 2.10.2.

EXERCISE 7.1.2. Figure 7.2 below illustrates the surgery description of a knot K in $S^3 = \partial B^4$. We see a canceling handle pair as in Figure 2.5 and the curve labeled K forms a ribbon knot as illustrated in Exercise 2.2.8. Prove that K has Alexander polynomial $t^2(1 - 3t + t^2)(1 - 3t^{-1} + t^{-2})$.

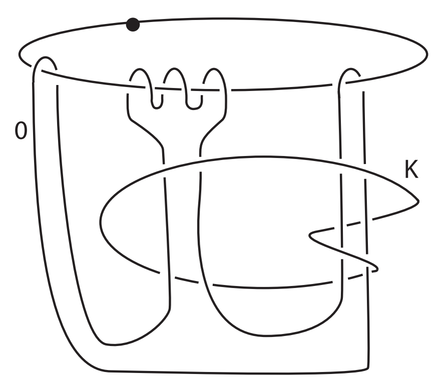

FIGURE 7.2. Surgery description for the exterior of a ribbon knot K with polynomial $t^2(1 - 3t + t^2)(1 - 3t^{-1} + t^{-2})$

Aside: The Alexander ideals over the integers

Since $\mathbb{Z}[t, t^{-1}]$ is not a principal ideal domain, the module $H_1(\tilde{X}(K))$ may not decompose as the direct sum of cyclic modules. We have the following however. Every finitely generated generated torsion module is presented by an $n \times m$ matrix with entries $\mathbb{Z}[t, t^{-1}]$, where $m \geq n$. There is an ideal A_i generated by all $(n - i) \times (n - i)$ minors. Every ideal in $\mathbb{Z}[t, t^{-1}]$ is contained in a unique maximal principal ideal, $< p_i(t) >$. This element is the i-Alexander invariant. It is given by the product $\prod_{j \leq n-i+1} f_j(t)$. Thus, the first Alexander invariant is the Alexander polynomial.

Exercise 7.1.3. Consider the $(n,1)$-cable of a knot K. It is formed as the curve $m + nl$ on $\partial X(K)$ and is denoted $K_{(n,1)}$. If K has Seifert matrix A, show the homology of the infinite cyclic cover of $K_{(n,1)}$ is presented by $A - t^n A^T$. (Do not do this as Seifert did, by finding a Seifert matrix for $K_{(n,1)}$. Rather, decompose the complement of $K_{(n,1)}$ using a torus and analyze the covers of the pieces.

7.2. Finite cyclic covers

Composing the map $\pi_1(X(K)) \to H_1(X(K)) \cong \mathbb{Z}$ with the surjection $\mathbb{Z} \to \mathbb{Z}/n$ yields a surjection $\pi_1(X(K)) \to \mathbb{Z}/n$. Associated to the kernel of this map there is an n–fold cover of $X(K)$ which we denote $\tilde{X}_n$ with group of deck transformations $\mathbb{Z}/n$.

7.2.1. The homology as a $\mathbb{Z}[\mathbb{Z}/n]$–module.

Theorem 7.2.1. As a $\mathbb{Z}[\mathbb{Z}/n]$–module (with $\mathbb{Z}/n$ generated by t), $H_1(\tilde{X}_n)$ is of the form $G \oplus \mathbb{Z}$ where $\mathbb{Z}$ is a trivial module and G is presented by the matrix $A - tA^T$.

Proof The cover $\tilde{X}_n$ can be constructed in much the same way as the infinite cyclic cover, using n copies of each space instead of an infinite collection. Much of the argument is as before. The only significant distinction is in the initial Mayer-Vietoris sequence. Paying attention to the H_0 terms now gives

$$0 \to H_2(\tilde{X}_n(K)) \to H_1(N^+) \oplus H_1(N^-) \to H_1(N) \oplus H_1(W) \stackrel{\mathrm{inc}_1 - \mathrm{inc}_2}{\to} H_1(\tilde{X}_n(K)) \to \mathbb{Z} \to 0.$$

This explains the factor of $\mathbb{Z}$ in the homology. □

7.3. The Milnor exact sequence

Our goal in this section is to prove the following theorem, and to introduce the Milnor exact sequence of the infinite cyclic cover, used here as a tool in proving the theorem.

Theorem 7.3.1. If $q = p^n$ is a prime power and $\pi\colon \tilde{X}_q \to X$ the q-fold cyclic cover, then $H_1(\tilde{X}_q) \cong G \oplus \mathbb{Z}$ where G is finite with order prime to p.

Proof We assume that a CW or simplicial structure has been placed on $X(K)$ and use the corresponding homology theory.

Let T denote the generating deck transformation of the infinite cyclic cover $\tilde{X}_\infty$. Milnor observed that there is the following exact sequence of chain complexes:

$$0 \to C_*(\tilde{X}_\infty) \stackrel{1-T_*}{\to} C_*(\tilde{X}_\infty) \stackrel{\pi_*}{\to} C_*(X) \to 0.$$

This yields an long exact sequence, which includes the following:

$$0 \to H_1(\tilde{X}_\infty) \overset{1-T_*}{\to} H_1(\tilde{X}_\infty) \overset{\pi_*}{\to} H_1(X) = \mathbb{Z} \to \mathbb{Z} \to 0, \tag{7.3.1}$$

since $H_2(X) = 0$ and $1 - T_*$ induces the 0 map on $H_0(\tilde{X}_\infty)$.

From this it follows that $1 - T_*$ is an isomorphism on $H_1(\tilde{X}_\infty)$.

Via π there is a natural map of $\pi_1(\tilde{X}_q)$ onto $\mathbb{Z}$ and a corresponding infinite cyclic cover. As a space this is homeomorphic to $\tilde{X}$, but now with the deck transformation replaced by T^q. (Stated otherwise, $\tilde{X}_q$ is the quotient of $\tilde{X}$ under the action of T^q.)

We now work with $\mathbb{Z}/p$-coefficients and have an exact sequence of chain complexes:

$$0 \to C_*(\tilde{X}_\infty, \mathbb{Z}/p) \overset{1-T_*^q}{\to} C_*(\tilde{X}_\infty, \mathbb{Z}/p) \overset{\pi_*}{\to} C_*(\tilde{X}_q, \mathbb{Z}/p) \to 0.$$

On homology this yields

$$H_2(\tilde{X}_q, \mathbb{Z}/p) \to H_1(\tilde{X}_\infty, \mathbb{Z}/p) \overset{1-T_*^q}{\to} H_1(\tilde{X}_\infty, \mathbb{Z}/p) \overset{\pi_*}{\to} H_1(\tilde{X}_q, \mathbb{Z}/p) \to \mathbb{Z}/p \to 0.$$

Since we are now working with $\mathbb{Z}/p$ coefficients and q is a power of p, $1 - T_*^q = (1 - T_*)^q$. Since $(1 - T_*)$ induces an isomorphism with $\mathbb{Z}$ coefficients, the same is true with $\mathbb{Z}/p$ coefficients, in which case $(1 - T_*)^q$ is an isomorphism.

It now follows that $H_2(\tilde{X}_q, \mathbb{Z}/p) = 0$ and that the map $H_1(\tilde{X}_q, \mathbb{Z}/p) \to \mathbb{Z}/p$ is an isomorphism. We already know that $H_1(\tilde{X}_q) = G \oplus \mathbb{Z}$. If tensoring with $\mathbb{Z}/p$ yields $\mathbb{Z}/p$, then G must be p–torsion. $\square$

7.4. Branched covers

7.4.1. Surfaces. Let F be a compact surface (perhaps with boundary) with designated points $P = \{p_1, \ldots, p_n\}$. Let X denote the complement of an open disk neighborhood of P and $\tilde{X}$ be a finite covering space of X. Consider the connected components of $\partial\tilde{X}$. Each one covers a component of ∂X via a map which, in complex coordinates, can be given as $z \to z^k$ for some $k \geq 1$. This map extends to a map of B^2, again given by $z \to z^k$. Thus, attaching disks to the boundary components of $\tilde{X}$ to form a closed surface $\tilde{F}$, there is a map $b\colon\ \tilde{F} \to F$ which is a covering space when restricted to $X - b^{-1}(P) \to F - P$. This is called a branched cover of F branched over the points P.

7.4.2. Riemann-Hurwitz formula. Given the surface F and the branching data for the cover, as next explained, the surface $\tilde{F}$ is easily determined. An example will illustrate.

EXAMPLE 7.4.1. Let F be a two sphere with three points designated, p_1, p_2, and p_3. There is a map of $\pi_1(F - P) \to \mathbb{Z}/6$ that sends the first meridian to 2, the second to 3, and the third to 1. (Note that the sum of the three meridians is 0 in homology.) In the associated covering, the first meridian has preimage with three components, and thus above p_1 there are three points in $\tilde{F}$. Above the second meridian there are two curves, so there are two points above p_2. Similarly, there is one point above p_3.

We can compute the Euler characteristic of F as follows. Select a triangulation of F in which the p_i are vertices. This lifts to a triangulation of $\tilde{F}$, with six simplices above each simplex in F, with the exception that there are two 0-simplices above p_1, three above p_2 and one above p_3. We have then, via a count of simplices, that $\chi(\tilde{F}) = 6\chi(F) - (4 + 3 + 5) = (6)(2) - 12 = 0$. Therefore, this 6–fold branched cover over S^2 is homeomorphic to T^2.

This approach to computing the Euler characteristic of the cover generalizes to give a general formula for an N–fold branched cover $\tilde{F}$ over F.

THEOREM 7.4.2. With the notation used above $\chi(\tilde{F}) = N\chi(F) - \sum d_i$, where d_i denotes the deficiency of the cover at each branch point.

EXERCISE 7.4.3. Let $p : \tilde{F} \to F$ be a connected N–fold cyclic branched cover of a surface of genus g with n branch points, in which all meridians go to generators of $\mathbb{Z}/N$. Compute the genus of $\tilde{F}$. Examine the question: for which g, n, and N does such a cover exist?

EXERCISE 7.4.4. Let F be a disk with 3 designated points. Let $h\colon \pi_1(F - P) \to \mathbb{Z}/6$ send the meridians to $1, 1$,and 2. There is a corresponding 6–fold branched cover, $\tilde{F}$, which can be described as a surface of genus g with l open disks removed. Find g and l.

7.4.3. Branched covers of S^3 over knots. Let $\tilde{X} \to X(K)$ be a finite cover of a knot complement. Attach to each boundary component of $\tilde{X}$ (each of which is homeomorphic to T^2) a solid torus, $V = S^1 \times B^2$, identifying each meridian of $S^1 \times B^2$ with a preimage of a meridian of K. Each meridian of V, a curve of the form $x \times \partial B^2$, maps to a meridian of K via a covering map. We can extend this to a map of V, sending each disk $x \times B^2$ to a normal disk to K, via a mapping of the form $z \to z^k$ for some $k \geq 0$ in complex coordinates. If we do this to each component of the boundary of $\tilde{X}$ we build a closed manifold $\bar{X}$ with a map $\pi\colon \bar{X} \to S^3$. There is an inclusion of $\tilde{X}$ into $\bar{X}$, and restricted to $\tilde{X}$, π is the original cover of $X(K)$.

EXAMPLE 7.4.5. **Cyclic Branched Covers of Knots** The most important example for us is the case of the n–fold cyclic cover $\tilde{X}_n$ of $X(K)$. The boundary of $\tilde{X}_n$ is a single torus, and on each preimage of a meridian of K, the map is a cyclic n–fold cover of the meridian. Thus, we attach a single solid torus to $\tilde{X}_n$ to build $\bar{X}_n$. On the core of the solid torus (the branched set of the map $\pi\colon \bar{X}_n \to S^3$) is mapped homeomorphically to K.

EXERCISE 7.4.6. Suppose that $\rho\colon \pi_1(S^3 - K) \to S_6$ is a homomorphism. This map determines a 6–fold cover of $X(K)$ and thus a 6–fold branched cover of S^3 branched over K. In each of the following situations, determine the number of branch curves in $\bar{M}$, the degree of the restriction of the branched cover to each component of the branch set (as a map to K), and the covering map restricted to a normal disk to a point on each of the components of the branch set.
(A) $\rho(m) = (12)$, $\rho(l) = 1$.
(B) $\rho(m) = (12)$, $\rho(l) = (345)$.
(C) $\rho(m) = (12)$, $\rho(m) = (12)$, $\rho(l) = (34)(56)$.
(D) $\rho(m) = (123)\rho(l) = (123)(456)$.

7.4.4. Homology of the cyclic branched covers. In Theorem 7.2.1, We have seen that the homology of the n-fold cyclic cover $\tilde{X}_n$ of a knot complement splits off a $\mathbb{Z}$-summand, generated by the preimage of a meridian. This element is killed by the attachment of the solid torus in forming the branched cover $\bar{X}_n$. Combining this with previous work, we have the following.

THEOREM 7.4.7. The homology of the n–fold cyclic branched cover $\bar{X}_n$ of a knot is presented as a $\mathbb{Z}[\mathbb{Z}/n]$–module by the matrix $A - tA^T$, where t is a generator of $\mathbb{Z}/n$.

7.4.5. The homology as a $\mathbb{Z}$–module. Suppose that t generates $\mathbb{Z}/n$ (so that $t^n = 1$) and that $T\colon \mathbb{Z}[t]^{2g} \to \mathbb{Z}[t]^{2g}$ is given by the matrix $\sum_{i=0,n-1} A_i t^i$. Then as a $\mathbb{Z}$–module $\mathbb{Z}[t]^{2g}$ is isomorphic to $\oplus_n \mathbb{Z}^{2g}$ The action of T is given by $T\colon (v, 0, 0, \ldots, 0) \to A_0 v, A_1 v, \ldots, A_{n-1} v)$. Similar descriptions can be found for the action on T on $(0, v, 0, \ldots, 0)$ and so on.

Applying this to the the $\mathbb{Z}[\mathbb{Z}/n]$ presentation of $H_1(\bar{X}_n)$ given by $A - tA^T$ we have the following [**?**, 8.D.7]:

THEOREM 7.4.8. The abelian group $H_1(\bar{X}_n)$ has presentation given by the $2gn \times 2gn$ matrix

$$P_1 = \begin{pmatrix} A^T & -A & 0 & \cdots & 0 & -A \\ 0 & A^T & -A & \cdots & 0 & 0 \\ 0 & 0 & A^T & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ -A & 0 & 0 & \cdots & 0 & A^T \end{pmatrix}. \tag{7.4.1}$$

EXAMPLE 7.4.9. Consider the case of the 2–fold cover. The presentation is given by

$$\begin{pmatrix} A^T & -A \\ -A & A^T \end{pmatrix}.$$

Adding the first row to the second, and then subtracting the first column from the second yield the presentation matrix

$$\begin{pmatrix} A^T & -A^T - A \\ A^T - A & 0 \end{pmatrix}.$$

Since $A^T - A$ is invertible, with determinant 1, we see that $H_1(\bar{X}_2)$ is presented by $A + A^T$. In particular, the order of $H_1(\bar{X}_2)$ is $|\det(A + A^T)| = |\Delta_K(-1)|$.

In a similar manner with matrix manipulations, it is possible to get a presentation for $H_1(\bar{X}_n)$ by a symmetric $2g(n-1) \times 2g(n-1)$ matrix. First add all the block rows to the bottom row, perform the appropriate column operations, and then use the fact that $A - A^T$ is invertible to eliminate the last (block) row and the last column; we get:

$$P_2 = \begin{pmatrix} -(A+A^T) & A & \cdots & \cdots & 0 & 0 \\ A^T & -(A+A^T) & A & \cdots & 0 & 0 \\ 0 & A^T & -(A+A^T) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & A^T & -(A+A^T) & A \\ \vdots & \vdots & \cdots & \cdots & A^T & -(A+A^T) \end{pmatrix}. \tag{7.4.2}$$

7.5. A simple presentation of $H_1(\bar{X}_n)$.

Seifert showed that the above matrix presentations for $H_1(\bar{X}_n)$ can be further reduced to be square with $2g$ generators. There are several proofs available in the literature, all based on matrix manipulation. Here we give a more algebraic approach.

THEOREM 7.5.1. Let A be a Seifert matrix for K. Define $E = A - A^T$ and $\Gamma = E^{-1}A$. Then $H_1(\bar{X}_n)$ is presented by the matrix $\Gamma^n - (\Gamma - 1)^n$

Proof

The homology of $\bar{X}_n$ is presented as a $\mathbb{Z}[\mathbb{Z}/n]$ module by $A - tA^T$, where t is here viewed as the generator of the cyclic group $\mathbb{Z}/n$. We denote this group as H_n.

Put otherwise,

$$H_n = Z^{2g}[t]/ < t^n - 1, A - tA^T > .$$

Here $\mathbb{Z}^{2g}[t]$ is a $\mathbb{Z}[t]$–module, and the quotient is by the images of the transformations $t^n - 1$ and $A - tA^T$. Henceforth we view A and A^T as transformations and note that both arise as transformations of $\mathbb{Z}^{2g} \subset \mathbb{Z}^{2g}[t]$.

We have that E is invertible acting on $\mathbb{Z}^{2g}$ since $\det(E) = 1$ so we can multiply $A - tA^T$ by E^{-1} without changing its image. This yields $\Gamma = E^{-1}A$ and we have:

$$H_n = \mathbb{Z}^{2g}[t]/ < t^n - 1, \Gamma - t(\Gamma - 1) > .$$

It follows that, acting on H_n, $\Gamma^n = (\Gamma - 1)^n$. Let $G = \Gamma - 1$, so $(G + 1)^n - G^n$ acts trivially. This equation can be expanded, and one finds that the identity 1 can be expressed as a polynomial in G with 0 constant term, and thus $G = \Gamma - 1$ is an invertible transformation of H_n. Call its inverse $(\Gamma - 1)^*$, but note that this is not a matrix, but rather a transformation of H_n. Note too that the inverse is in fact a transformation of $\mathbb{Z}^{2g}$.

Multiplying $\Gamma - t(\Gamma - 1)$ by $(\Gamma - 1)^*$, we see that $\Gamma(\Gamma - 1)^* - t$ acts trivially on H_n. From this it follows that the inclusion

$$F_1 : \mathbb{Z}^{2g}/ < \Gamma^n - (\Gamma - 1)^n > \;\longrightarrow\; \mathbb{Z}^{2g}[t]/ < t^n - 1, \Gamma - t(\Gamma - 1) >$$

is well-defined and surjective.

Any element of $\mathbb{Z}^{2g}[t]$ can be written as a sum of elements of the form $t^k v$, where $v \in \mathbb{Z}^{2g}$. Define a group homomorphism

$$F_2 : \mathbb{Z}^{2g}[t]/ < t^n - 1, \Gamma - t(\Gamma - 1) > \;\longrightarrow\; \mathbb{Z}^{2g}/ < \Gamma^n - (\Gamma - 1)^n >$$

by $F_2(t^k v) = (\Gamma(\Gamma-1)^*)^k(v)$. This is clearly surjective and the composition F_2F_1 is the identity. In conclusion,

$$H_n \cong \mathbb{Z}^{2g}/ < \Gamma^n - (\Gamma-1)^n > .$$

□

EXAMPLE 7.5.2. In applying the formula $\Gamma^n - (\Gamma-1)^n$ it is often useful to diagonalize the matrix Γ before taking their powers. As an interesting example that will be of importance later (see 7.10.3), consider the metabolic form

$$\begin{pmatrix} 0 & a+1 \\ a & 1 \end{pmatrix}.$$

For a knot with this form, a presentation for $H_1(X_n)$ is given by

$$\begin{pmatrix} \alpha & (1+(-1)^n)\alpha/(2a+1) \\ 0 & \alpha \end{pmatrix},$$

where $\alpha = (a+1)^n - a^n$. Notice that if n is odd this group is a double (something that is actually true for all odd-fold cyclic branched covers of arbitrary knots) and if n is even the homology is never a double unless a is 0 or -1.

REMARK 7.5.3. The first homology of the 0-surgery on S^3 along a knot is $\mathbb{Z}$. If $\bar{X}$ denotes the n-fold cyclic cover of S^3 branched over K, and $\bar{M}$ denotes the n-fold cyclic cover of the 0-surgery on S^3 along K, then $H_1(\bar{M}) \cong H_1(\bar{X}) \oplus \mathbb{Z}$. Equivalently, if $\tilde{X}$ is the n-fold cyclic cover of the knot exterior $X(K)$, then $H_1(\bar{M}) \cong H_1(\tilde{X})$. See 7.12.5 for more on covers of the 0-surgery.

EXAMPLE 7.5.4. An easy calculation shows that the 6-fold cyclic branched cover of the trefoil knot, with Seifert form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

has $H_1(X_6) = \mathbb{Z} \oplus \mathbb{Z}$.

One can show that for an arbitrary knot, $H_1(\bar{X}_n)$ is infinite if and only if $\omega_n = e^{2\pi i/n}$ is a root of the Alexander polynomial. In short, $t^n - 1$ induces on isomorphism on $H_1(\tilde{X}, \mathbb{Q})$, if $\bar{X}_n$ is a rational homology sphere, and in this case $t^n - 1$ is invertible in $\mathbb{Q}[t, t^{-1}]/|H_1(\tilde{X})|$, where the order in the denominator is as a $\mathbb{Q}[t, t^{-1}]$ module. Finally, this order is given by $\Delta_K(t)$. Theorem 7.5.5 below gives a stronger statement, originally proved by Fox.

THEOREM 7.5.5. The first homology of the n-fold cylcic cover of S^3 branched over a knot, $H_1(\bar{X}_n)$, is finite if and only no root of its Alexander polynomial is an n-th root of unity. In the case that $H_1(\bar{X}_n)$ is finite, its order is given by

$$|H_1(\bar{X}_n)| = |\prod_{i=1,n-1} \Delta_K(\omega_n^i)\ |,$$

where ω_n is a primitive root of unity.

Proof The proof is notationally complex and we won't give the details. But with the following special case should indicate to the reader the key idea. Consider the $\mathbb{Z}[\mathbb{Z}/n]$ module $\mathbb{Z}[\mathbb{Z}/n]/ < f(t) >$ where $f(t) \in \mathbb{Z}[\mathbb{Z}/n]$. This is a $\mathbb{Z}$–module, that is, an abelian group. What is its order?

Another way of asking the same question is the following. Multiplication by $f(t)$ defines a linear transformation of the free abelian group $\mathbb{Z}[\mathbb{Z}/n]$. What is its determinant?

If we tensor with $\mathbb{F} = \mathbb{Q}[\omega_n]$ (or if you prefer with $\mathbb{C}$) then $\mathbb{F}[\mathbb{Z}/n]$ splits into the eigenspaces of the action of multiplication by t. These are spanned by the elements $\{1 + t + \cdots + t^{n-1}, 1 + \omega_n t + \omega_n^2 t^2 + \cdots, \ldots, 1 + \omega^{n-1} t + \cdots\}$. Multiplication by $f(t)$ preserves each of these eigenspaces, multiplying the first by $f(1)$, the second by $f(\omega_n)$, and so on. Thus, the product of these gives the determinant, as desired. □

7.6. Linking forms

For a closed oriented 3–manifold M there is a nonsingular pairing Torsion$(H_1(M) \times H_1(M)) \to \mathbb{Q}/\mathbb{Z}$, denoted $(x, y) \to \mathrm{lk}(x, y)$. By nonsingular we mean that the induced map $H_1(M) \to \mathrm{Hom}(H_1(M), \mathbb{Q}/\mathbb{Z})$ is an isomorphism. We have seen a purely algebraic discussion of linking forms in Section 5.5. We now give a geometric description of this pairing, and also the algebraic definition in the case that M is a rational homology sphere.

Suppose that x and y are torsion classes in $H_1(M)$. Represent each by disjoint embedded curves, α and β. We have that $n\alpha = \partial c$ for some singular chain c. We can assume that β is transverse to the cells of c and then compute the signed intersection number, $\beta \cap c$. We define $\mathrm{lk}(x, y) = \frac{1}{n}(\beta \cap c) \in \mathbb{Q}/\mathbb{Z}$. One can give geometric arguments that this is well-defined and symmetric. Nonsingularity is harder, depending on Poincare duality.

(As an alternative to singular homology, one can use simplicial homology, letting α and c reside in one triangulation and β reside in the dual cell decomposition. In this way defining the intersection number can be done more formally.)

As an algebraic alternative, consider the exact sequence of groups:

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}.$$

This induces a long exact sequence on cohomology with connecting homomorphism (a "Bockstein map") $\delta : H^1(M, \mathbb{Q}/\mathbb{Z}) \to H^2(M, \mathbb{Z})$. In the case that $H_1(M, \mathbb{Q}) = 0$, δ is an isomorphism. But $H^1(M, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(H_1(M), \mathbb{Q}/\mathbb{Z})$ and by Poincare duality $H^2(M, \mathbb{Z}) \cong H_1(M)$. Thus, we have an isomorphism $\mathrm{Hom}(H_1(M), \mathbb{Q}/\mathbb{Z}) \cong H_1(M)$. This can be used to define the linking pairing.

Example 7.6.1. For p and q relatively prime integers, $p \geq 0$, the lens space $L(p, q)$ is defined to the the union of two solid tori, $S^1 \times B^2) \cup S^1 \times B^2$ so that the meridian of one is identified with $qm + pl$ on the other. (If $p = 0$ we consider only the case $q = 1$, so that $L(0, 1) \cong S^1 \times S^2$. We have $H_1(L(p, q)) \cong \pi_1(L(p, q)) \cong \mathbb{Z}/p$.

Exercise 7.6.2. The core of one of the solid tori represents a generator of $H_1(L(p, q))$. Show that $\mathrm{lk}(x, x) = \frac{q}{p} \in \mathbb{Q}/\mathbb{Z}$.

7.7. The rational Blanchfield pairing

If $\tilde{X}$ is the infinite cyclic cover of $X(K)$ we know that $H_1(\tilde{X}, \mathbb{Q})$ is a torsion $\mathbb{Z}[t^{\pm 1}]$–module. Given an element $x \in H_1(\tilde{X}, \mathbb{Q})$ we thus have $p(t)x = 0$ for some polynomial p. If α is a closed curve representing x, then $p(t)\alpha = \partial c$ for some chain c. We define the Blanchfield pairing $\beta(x, y)$ by selecting a curve β representing y and setting

$$\beta(x, y) = \sum_k (\beta \cap c) t^k \in \mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}].$$

In defining the intersection number one must take into account that the chains are rational, in the obvious way. The sum is finite by compactness.

The following result holds:

Theorem 7.7.1. The Blanchfield pairing is skew linear and symmetric:

$$\beta(x, y) = \overline{\beta(y, x)}$$

and

$$\beta(f(t)x, g(t)y) = f(t)\bar{g}(t)\beta(x, y),$$

where $\bar{h}(t) = h(t^{-1})$ for a polynomial h.

A deep theorem of Trotter [**?**] states that the Seifert forms for two knots are S–equivalent if and only if they have the same Blanchfield pairing.

7.8. Finite Cyclic branched Covers of B^4

For any knot K in S^3 there is a connected properly embedded surface F in B^4 with boundary K; for example, a pushed in Seifert surface for K is one such. For each n one can construct the n-fold cyclic cover of $B^4 - F$ and the corresponding branched cover. We denote these by W_n and $\bar{W}_n$ respectively. Of course K can bound many such surfaces, and so most invariants of W_n and $\bar{W}_n$ (for instance the homology groups) are not invariants of K. However, as we will see later, certain invariants, such as Witt class invariants of the intersection form on $H_2(\bar{W}_n, \mathbb{Q})$, are independent of the choice of surface. Furthermore, we will see that in the case that if F is a slice disk, the homology of $\bar{W}_n$ is quite constrained, and these constraints lead to concordance invariants.

Once we relate the homology of $\bar{W}_n$ to the Seifert form, the connection to the algebraic invariants of the last chapter will be clear, as will be the fact that concordance invariants have been obtained. However, there are several reasons to carry out the homological arguments; the techniques give results in other settings such as slicing problems in other 4-manifolds and, most important, similar arguments are used to derive the results of Casson and Gordon which we study in the next chapter.

Our study of W_n and $\bar{W}_n$ will, as in the knot complement case, be based on a study of the homology of the infinite cyclic cover, W_∞, via the Milnor sequence. The information we gather concerning W_∞ will be of use to us in the next section.

Assume now that F is a connected properly embedded surface of genus g in B^4 with boundary a knot K. We will limit ourselves to prime power covers of the 4-ball, W_{p^k} and $\bar{W}_{p^k}$. Alexander duality implies that $H_1(B^4 - F) = \mathbb{Z}$, $H_2(B^4 - F) = \mathbb{Z}^{2g}$, and $H_3(B^4 - F) = H_4(B^4 - F) = 0$. These facts, along with a simple analysis of the H_0 terms leads to the following exact sequences arising from the Milnor exact sequence.

$$0 \to H_3(W_\infty) \xrightarrow{t-1} H_3(W_\infty) \to 0$$

$$0 \to H_2(W_\infty) \xrightarrow{t-1} H_2(W_\infty) \to \mathbb{Z}^{2g} \to H_1(W_\infty) \xrightarrow{t-1} H_1(W_\infty) \to 0$$

An immediate consequence is that $t - 1$ on homology is:

a. surjective on $H_1(W_\infty)$,
b. injective on $H_2(W_\infty)$,
c. an isomorphism on $H_3(W_\infty)$.

To begin our analysis we switch to working with coefficients in a field $\mathbb{F}$ so that we can apply the fundamental theorem of finitely generated modules over a PID: each such module is the direct sum of modules of the form $\mathbb{F}[t,t^{-1}]$, and $\mathbb{F}[t,t^{-1}]/<p(t)>$, where the $p(t)$ are powers of irreducible polynomials. In particular, each such module is the direct sum of $\mathbb{F}[t,t^{-1}]^k$ with modules of the form $\mathbb{F}[t,t^{-1}]/<(t-1)^j>$, and $\mathbb{F}[t,t^{-1}]/<p(t)>$, where $p(t)$ is relatively prime to $t-1$.

LEMMA 7.8.1. We have the following.

a. $H_1(W_\infty,\mathbb{F})$ is the direct sum of modules of the form $\mathbb{F}[t,t^{-1}]/<p(t)>$, where $p(t)$ is relatively prime to $t-1$; furthermore, $t-1$ is an isomorphism on $H_1(W_\infty,\mathbb{F})$.
b. $H_2(W_\infty,\mathbb{F})$ is the direct sum of $\mathbb{F}[t,t^{-1}]^{2g}$ and modules of the form $\mathbb{F}[t,t^{-1}]/<p(t)>$, where p is relatively prime to $t-1$.
c. $H_3(W_\infty,\mathbb{F})$ is the direct sum of modules of the form $\mathbb{F}[t,t^{-1}]/<p(t)>$, where $p(t)$ is relatively prime to $t-1$.

Proof a) The first statement follows immediately from the fact that $t-1$ is surjective. Notice that on modules of this form $t-1$ is also injective.
b) We make two simple observations: $t-1$ has cokernel $\mathbb{F}^{2g}$ and is injective as a map of $H_2(W_\infty,\mathbb{F})$. From this it follows that $H_2(W_\infty,\mathbb{F})$ has no summands of the form $\mathbb{F}[t,t^{-1}]/<(t-1)^j>$, and has exactly $2g$ summands of the form $\mathbb{F}[t,t^{-1}]$.
c) This follows from the fact that $t-1$ is an isomorphism on $H_3(W_\infty,\mathbb{F})$. □

With this lemma, we can now consider the Milnor sequence for W_{p^k}, where the deck transformation is given by $t^{p^k}-1$. We begin in with the case of $\mathbb{Z}/p$ coefficients.

LEMMA 7.8.2. We have:

a. $H_1(W_{p^k},\mathbb{Z}/p)=\mathbb{Z}/p$,
b. $H_2(W_{p^k},\mathbb{Z}/p)=(\mathbb{Z}/p[t,t^{-1}]/<(t-1)^{p^k}>)^{2g}=(\mathbb{Z}/p)^{p^k 2g}$, with the first equality as a $\mathbb{Z}/p[t,t^{-1}]$ module and the second as a $\mathbb{Z}/p$ module.
c. $H_3(W_{p^k},\mathbb{Z}/p)=0$.

Proof Recall that with $\mathbb{Z}/p$ coefficients $t^{p^k}-1=(t-1)^{p^k}$. With this the results follow immediately from the previous lemma. □

We can now consider the rational case.

LEMMA 7.8.3. We have:

a. $H_1(W_{p^k},\mathbb{Q})=\mathbb{Q}$,

b. $H_2(W_{p^k}, \mathbb{Q}) = (\mathbb{Q}[t, t^{-1}]/ < t^{p^k} - 1 >)^{2g} = \mathbb{Q}^{p^k 2g}$, with the first equality as a $\mathbb{Q}[t, t^{-1}]$ module and the second as a $\mathbb{Q}$ module.

c. $H_3(W_{p^k}, \mathbb{Q}) = 0$.

Proof To prove (a), first, recall that the rank of the $\mathbb{Z}/p$ homology bounds the dimension of the $\mathbb{Q}$ homology. Hence, $\dim(H_1(W_{p^k}, \mathbb{Q})) \leq 1$. Considering the H_0 term shows that the dimension is at least 1, so we have $H_1(W_{p^k}, \mathbb{Q}) = \mathbb{Q}$. From this it follows that $t^{p^k} - 1$ is injective on $H_1(W_\infty, \mathbb{Q})$, so in the decomposition of $H_1(W_\infty, \mathbb{Q})$ as the sum of modules of the form $\mathbb{Q}[t, t^{-1}]/ < p(t) >$, each $p(t)$ is prime to $t^{p^k} - 1$ and hence $t^{p^k} - 1$ is an isomorphism on $H_1(W_\infty, \mathbb{Q})$.

Next we consider $H_2(W_{p^k}, \mathbb{Q})$, and observe that $\dim(H_2(W_{p^k}, \mathbb{Q})) \leq p^k 2g$. On the other hand, $H_2(W_{p^k}, \mathbb{Q})$ is the cokernel of the map $t^{p^k} - 1$ on $H_2(W_\infty, \mathbb{Q}) = (\mathbb{Q}[t, t^{-1}])^{2g} \oplus_i \mathbb{Q}[t, t^{-1}]/ < p_i(t) >$. The quotient of the free part contributes $p^k 2g$ to the dimension already, so there must be no remaining cokernel. In particular, each $p_i(t)$ is prime to $t^{p^k} - 1$, so $t^{p^k} - 1$ induces an isomorphism on these factors. Condition (c) follows from condition (c) of the previous lemma. □

We conclude these homological observations by considering the consequence for the branched cover. Here is the result.

THEOREM 7.8.4. If $\bar{W}_{p^k}$ is the p^k-fold cyclic cover of B^4 branched over a genus g properly embedded connected surface with connected boundary, then

a. $H_1(\bar{W}_{p^k}, \mathbb{Q}) = 0$.

b. $H_2(\bar{W}_{p^k}, \mathbb{Q}) = (\mathbb{Q}[t, t^{-1}]/ < N >)^{2g}$ where $N = (t^{p^k} - 1)/(t - 1)$. In particular, $H_2(\bar{W}_{p^k}, \mathbb{Q}) = \mathbb{Q}^{(p^k-1)(2g-1)}$.

c. $H_3(\bar{W}_{p^k}, \mathbb{Q}) = 0$.

Proof The branched cover is constructed by attaching a copy of $F \times B^2$ to W_{p^k} along $F \times S^1$. A Mayer-Vietoris argument applies to show that the homology of $\bar{W}_{p^k}$ is a quotient of the homology of W_{p^k}.

Result (c) follows immediately, and (a) follows from noting that the meridian to the surface generates $H_1(\bar{W}_{p^k}, \mathbb{Q})$, and this class dies in $H_1(F \times B^2, \mathbb{Q})$. The most subtle point is result (b).

Notice that the image of the transfer map $\tau : H_2(B^4 - F, \mathbb{Q}) \to H_2(W_{p^k}, \mathbb{Q})$ contains the elements represented by $(t^{p^k} - 1)/(t - 1)$ in each $\mathbb{Q}[t, t^{-1}]/ < (t^{p^k} - 1) >$ summand of $H_2(W_{p^k}, \mathbb{Q}) = (\mathbb{Q}[t, t^{-1}]/ < t^{p^k} - 1 >)^{2g}$, since each of these is invariant. A dimension count shows that these must actually generate the image of the transfer. Next observe that the image of the transfer is carried by the image of the transfer restricted to the boundary since $H_2(F \times S^1, \mathbb{Q})$ maps onto $H_2(B^4 - F, \mathbb{Q})$). Finally, these elements are precisely the ones killed upon adding in the branch set. $\square$

7.8.1. Branching over a Seifert surface: the intersection form. Let F be a Seifert surface for K, and let F^* be the properly embedded surface in B^4 constructed by pushing F radially into B^4, fixing its boundary. In this case, the intersection form of the branched cover, $\bar{W}_{p^k}$, can be explicitly computed in terms of the Seifert matrix V associated to F. These results were first obtained by Kauffman and Taylor [**?**] in the case of $p^k = 2$ and in the general case by Kauffman and Neumann [**?**]. Before stating the main result, we remind the reader of the definition of the intersection form of a 4-manifold, and of the generalization in the case that a group G acts on the manifold.

Let W be a fixed compact 4-manifold. The following sequence defines a pairing on $H_2(W)$, and hence induces one on $H_2(W)/\text{torsion}$.

$$H_2(W)\times H_2(W) \to H_2(W)\times H_2(W,\partial W) \to H_2(W)\times H^2(W) \to H_2(W)\times \text{Hom}(H_2(W)) \to \mathbb{Z}$$

In the case that ∂W is a rational homology sphere, the pairing is nondegenerate—it induces an injection $H_2(W) \to \text{Hom}(H_2(W))$ though perhaps not an isomorphism. (To see this, work with rational coefficients, so that $H_2(W) \to H_2(W, \partial W)$ is an isomorphism.) We denote this pairing by $x \times y \to < x, y >$.

There is a geometric interpretation of the intersection form. Given classes x and y in $H_2(W)$, represent them by disjoint transverse surfaces (or cycles) and compute the algebraic intersection number.

Suppose now that a group G acts on W, where we assume groups act on spaces on the right. Then we define a pairing of the $\mathbb{Z}[G]$ modules $H_2(W)/\text{torsion} \times H_2(W)/\text{torsion} \to \mathbb{Z}[G]$ by $(x, y) \to \sum_{g\in G} < x, yg > g$, and denote the form by $(x, y) \to < x, y >_G$. Restricting to the case in which G is abelian, we have that the $\mathbb{Z}[G]$-valued intersection form satisfies $< g_1x, g_2y >_G = g_1g_2^{-1} < x, y >_G$.

At this point we will restrict ourselves to the case of immediate interest to us; we will assume that G is a finite cyclic group, say of order n, and also switch to rational coefficients. In this case the map $\mathbb{Z}[G] \to \mathbb{Q}[\omega]$, where ω is a primitive n-root of unity yields a form $H_2(W, \mathbb{Q}) \times H_2(W, \mathbb{Q}) \to \mathbb{Q}[\omega]$. This form now may be degenerate, but one can check that the induced form on $H_2(W, \mathbb{Q}) \otimes_{\mathbb{Q}[G]} \mathbb{Q}(\omega)$ is nonsingular, viewed as a form on $\mathbb{Q}(\omega)$ vector spaces.

There is an another definition of the $\mathbb{Q}[\omega]$ intersection form just described. Let W be any compact 4-manifold with a homeomorphism T satisfying $T^n=1$. Then $H_2(W,C)$ splits into ω^i eigenspaces of the action of T, where i ranges from 0 to $n-1$. In fact, the same is true with $\mathbb{Q}(\omega)$ coefficients; here is an easy proof. Using the action of T, $H_2(W,\mathbb{Q}(\omega))$ is a $\mathbb{Q}(\omega)[t,t^{-1}]$ module, and since W is compact, it must be a torsion module. Since $H_2(W)$ is annihilated by T^n-I, all factors in its cyclic decomposition can be assumed to be of the form $\mathbb{Q}(\omega)[t,t^{-1}]/<p(t)>$ where $p(t)$ is a prime factor of t^n-1. These factors are all of the form $t-\omega^i$ for some i in the given range. This gives the desired decomposition into eigenspaces. We denote the ω-eigenspace by $H_2(W)_\omega$. The intersection form on $H_2(W,\mathbb{Q}(\omega))$ can be restricted to $H_2(W)_\omega$. The form that results is n times the $\mathbb{Q}[\omega]$–valued form defined previously. (The map $H_2(W,\mathbb{Q})\to H_2(W,\mathbb{Q}(\omega))$ defined by $x\to\sum_{i=0}^{n-1}\omega^{-i}T^i(x)$ induces an isomorphism $H_2(W,\mathbb{Q})\otimes_{\mathbb{Q}[G]}\mathbb{Q}(\omega)\to H_2(W)_\omega$.)

For clarity we summarize these previous observations. Both of the homology groups $H_2(W,\mathbb{Q}(\omega))$ and $H_2(W,\mathbb{Q})\otimes_{\mathbb{Q}[\mathbb{Z}/n]}\mathbb{Q}(\omega)$ are $\mathbb{Q}(\omega)$ vector spaces. The ω eigenspace of the $\mathbb{Z}/n$ action on $H_2(W,\mathbb{Q}(\omega))$ is isomorphic to $H_2(W,\mathbb{Q})\otimes_{\mathbb{Q}[\mathbb{Z}/n]}\mathbb{Q}(\omega)$. Each has an intersection form, the first the restriction of the standard intersection form and the second the twisted intersection form defined previously. Under the isomorphism the two forms correspond (up to a constant multiple).

Theorem 7.8.5. If $\bar{W}_{p^k}$ is the p^k-fold branched cover of B^4 branched over a pushed in Seifert surface for a knot K with Seifert matrix A, then $H_2(\bar{W}_{p^k},\mathbb{Q}(\omega))_\omega$ is a $2g$ dimensional $\mathbb{Q}(\omega)$ vector space, where ω is a primitive p^k-root of unity. There is a basis corresponding to the generators of $H_1(F)$, and under this correspondence the $\mathbb{Q}(\omega)$ intersection form on $H_2(\bar{W}_{p^k})$ is given by the matrix $(1-\omega)A+(1-\bar{\omega})A^T$.

Proof Let $\hat{F}$ denote the (3-dimensional) trace of the radial isotopy carrying F to its pushed-in (with boundary fixed) copy F^*. As in the construction of the cyclic covers of S^3-K based on piecing together copies of S^3-F, the n-fold cyclic cover of B^4-F^* proceeds by piecing together copies of $B^4-N(\hat{F})$.

A Mayer-Vietoris argument now applies with the cover W_{p^k} described as the union of W_I andW_{II}, where W_I is the disjoint union of n copies of $B^4-N(\hat{F})$ and W_{II} is that of n-copies of $N(\hat{F})=\hat{F}\times I\cong F\times I\times I$. The intersection deformation retracts to n copies of $F\amalg F$. Noting that $B^4-N(\hat{F})$ is contractible, over $\mathbb{Z}[\mathbb{Z}/p^k]$ we have the following relevant portion of the Mayer-Vietoris sequence.

$$0\to H_2(W_{p^k})\to H_1(F)\oplus H_1(F)\to H_1(F).$$

Hence, for each generator of $H_1(F)$ there is a $\mathbb{Z}[\mathbb{Z}/p^k]$ generator of $H_2(W_{p^k})$. A cycle representing that generator is constructed as follows. Fix a distinguished copy of $N(\hat{F})$, $N(\hat{F})_0$, in the decomposition of W_{p^k} and pick a representative cycle x for a class in $H_1(F)$. Then $x \times [-1,1]$ embeds in $N(\hat{F})_0$, with the $[-1,1]$ factor corresponding to the normal direction to $\hat{F}$. After the identification, $x \times \{-1\}$ and $x \times \{1\}$ bound surfaces (or for that matter cones) in successive copies of $B^4 - N(\hat{F})$. From here the derivation of the relationship of the intersection form of these classes to the Seifert form is clear, achieving the precise formula involves some detailed but straightforward calculations.

Finally, we must consider the effect of adding the branch set to form $\bar{W}_{p^k}$. Again a Mayer-Vietoris argument applies in that we are completing the space by adding a copy of $F \times B^2$ along $F \times S^1$. For each 1- cycle x in F there is a product cycle $x \times S^1$ in $F \times S^1$, and these generate $H_2(F \times S^1)$. These cycles are clearly null homologous in $F \times B^2$, and the Mayer-Vietoris sequence implies that $H_2(\bar{W}_{p^k})$ is obtained from $H_2(W_{p^k})$ by killing the images of these classes. But in the explicit construction of a class in $H_2(W_{p^k})$ from a class x in $H_1(F)$ of the previous paragraph, it is clear that the sum of that constructed class and its translates is precisely the class $x \times S^1$ in the kernel. (The successive cones cancel, leaving only the copies of $x \times [-1,1]$, whose union is $x \times S^1$. □

7.8.2. The integral intersection form and the homolgy of X_n. In the case that we branch the 4-ball over a pushed in Seifert surface, the fundamental group of the complement is isomorphic to $\mathbb{Z}$, and hence the finite branched cover is simply connected. It follows that $H_2(\bar{W}_n, \mathbb{Z})$ is free. From our previous analysis we find that $H_2(\bar{W}_n, \mathbb{Z}) = \mathbb{Z}^{2g(n-1)}$. A careful examination of our previous work shows that the integral intersection form is given by the matrix P_2 in 7.4.2. Recall that $\bar{X}_n$ denotes the n-fold cyclic cover of S^3 branched over K. The long exact sequence of $(\bar{W}_n, \bar{X}_n)$ gives the following presentation of $H_1(\bar{X}_n)$:

$$H_2(\bar{W}_n) \to H_2(\bar{W}_n, \bar{X}_n) \to H_1(\bar{X}_n) \to 0$$

We have by Poincare duality that $H_2(\bar{W}_n, \bar{X}_n)$ is isomorphic to $\mathrm{Hom}(H_2(\bar{W}_n), \mathbb{Z})$, and so we have the presentation

$$H_2(\bar{W}_n) \to \mathrm{Hom}(H_2(\bar{W}_n), \mathbb{Z}) \to H_1(\bar{X}_n) \to 0.$$

The first map is given by the matrix P_2 with respect to the given basis and its dual basis. Hence we have that P_2 presents $H_1(\bar{X}_n)$.

7.9. The infinite cyclic cover of $B^4 - F$

In this section we will just summarize the consequences of the work of the previous section as it relates specifically to the homology (and intersection form) of W_∞, with a focus on rational coefficients. Our first result was developed in the proof of Lemma 7.8.1 in the previous section.

LEMMA 7.9.1. We have:

a. As a $\mathbb{Q}[t,t^{-1}]$ module, $H_1(W_\infty,\mathbb{Q})$ is the finite direct sum of modules of the form $\mathbb{Q}[t,t^{-1}]/ < p(t) >$, where $p(t)$ is relatively prime to $t^{p^k}-1$ for all primes p and positive k.
b. $H_2(W_\infty,\mathbb{Q})$ is the finite direct sum of $\mathbb{Q}[t,t^{-1}]^{2g}$ and modules of the form $\mathbb{Q}[t,t^{-1}]/ < p(t) >$ where $p(t)$ is relatively prime to $t^{p^k}-1$ for all primes p and positive k.
c. $H_3(W_\infty,\mathbb{Q})$ is the finite direct sum of modules of the form $\mathbb{Q}[t,t^{-1}]/ < p(t) >$, where $p(t)$ is relatively prime to $t^{p^k}-1$ for all primes p and positive k.

EXERCISE 7.9.2. In Figure 7.2 of Exercise 7.1.2, we saw a surgery description of a knot K in S^3. The same picture may be used to represent B^4 with a canceling pair of 1 and 2-dimensional handles. The knot K in ∂B^4 bounds an obvious disk in B^4. The core of the 2-handle intersects this disk in ribbon singularities which are easily removed by pushing further into $\text{Int}(B^4)$. (See 3.5.) Thus it is seen that K is slice, in fact, ribbon.

Prove that in this example, $H_1(W_\infty,\mathbb{Q})$ is the cyclic module $\mathbb{Q}[t,t^{-1}]/\langle 1-3t+t^2\rangle$.

REMARK 7.9.3. Exercises 7.1.2 and 7.9.2 illustrate a ribbon knot construction with a specified Alexander polynomial $f(t)f(t^{-1})$ following the methods in Section 3 of [**?**]. The polynomial is normalized by multiplying with an appropriate power of t to eliminate negative powers and have a nontrivial constant term. Also see [**?**] and [**?**].

There is an intersection form on the free part of the $\mathbb{Q}[t,t^{-1}]$ module $H_2(W_\infty,\mathbb{Q})$, or, more precisely, on $H_2(W_\infty,\mathbb{Q})/\text{torsion}$. This quotient is isomorphic to $\mathbb{Q}[t,t^{-1}]^{2g}$. The pairing is best defined using duality as in the case of finite covers; the only difficulty is that $H_2(W_\infty,\partial W_\infty)$ is dual to $H^2_c(W_\infty)$, the second cohomology with compact support.

We will review this cohomology theory and the associated duality, but for now, we give the geometric description. Given classes x and y, their intersection number is computed by representing them by transverse surfaces or cycles and computing the algebraic intersection number of those representatives. This form is nonsingular; as in the case of finite covers, the proof depends on the fact that $H_1(W_\infty, \mathbb{Q})$ is torsion, only now over $\mathbb{Q}[t, t^{-1}]$, rather than $\mathbb{Z}$. Again, the details depend on the duality theory that must await the next chapter.

In the case that F is the pushed in Seifert surface for a knot K, the geometric description of a basis for $H_2(W_\infty, \mathbb{Q})$ as a $\mathbb{Q}[t, t^{-1}]$ module is much the same as in the case of a finite cover described in the previous section. The result is:

THEOREM 7.9.4. If A is the Seifert form for a Seifert surface F of a knot K, then the intersection form of $H_2(W_\infty, \mathbb{Q})$ is given by $(1-t)A + (1-t^{-1})A^T$.

7.10. Elementary slicing obstructions arising from X_n

The simplest means to deriving slicing obstructions for a knot based on X_n are to use the results of the previous sections along with the fact that for a slice knot there is a Seifert matrix which has a half dimensional diagonal block of $0's$. We will be seeking more intrinsic approaches to obstructions, but begin with two results for which the Seifert matrix based proof is immediate. We should note that although there are a number of advantages to the intrinsic approach, when it comes to doing explicit calculations one usually turns to algorithms which call upon the Seifert matrix.

7.10.1. Seifert matrix based results.

THEOREM 7.10.1. If K is slice and X_n is a rational homology sphere, then $|H_1(X_n)|$ is a square.

The above result also follows from the form the Alexander polynomial of a slice knot takes (see 4.3.2), along with the relationship between the Alexander polynomial of K and order of the first homology of a finite, cyclic cover of S^3 branched over K described in Theorem 7.5.5.

One can also use these formulas to derive obstructions to a knot being doubly null concordant.

THEOREM 7.10.2. If K is doubly null concordant, then $H_1(X_n) = H_1 \oplus H_1$ for some finitely generated abelian group H_1.

EXAMPLE 7.10.3. In the case of a knot with Seifert matrix

$$A = \begin{pmatrix} 0 & a+1 \\ a & 1 \end{pmatrix},$$

$H_1(X_2)$ is presented by the matrix $A + A^T$,

$$\begin{pmatrix} 0 & 2a+1 \\ 2a+1 & 2 \end{pmatrix}.$$

The homology group is isomorphic to $\mathbb{Z}/(2a+1)^2$ and hence does not split into a double unless $a = 0$ or -1. It follows that except in these two exceptional cases the knot is not doubly null concordant.

Recall that we mentioned before in 4.2.9 that the fact that a knot with such a matrix is not doubly null concordant does not follow solely from the fact that the Seifert form does not split over metabolic summands; we only know that such a splitting occurs for *some* Seifert form of a doubly null concordant knot. Also see 7.5.2.

7.10.2. Results based on the branched cover of B^4 branched over a slice disk. In order to determine the special properties of a cyclic branched cover $\bar{X}_n$ of S^3 branched over a slice knot K, we can consider the corresponding branched cover $\bar{W}_n$ of the 4-ball, branched over a slice disk B for K.

THEOREM 7.10.4. If K is a slice knot then $|H_1(\bar{X}_{p^k})|$ is a square for all prime powers p^k. In particular, the kernel of the inclusion $H_1(\bar{X}_{p^k}) \to H_1(\bar{W}_{p^k})$ is a subgroup of order the square root of $|H_1(\bar{X}_{p^k})|$, and this kernel is invariant under the action of the deck transformations on $\bar{X}_{p^k}$.

Proof We have seen that $\bar{W}_{p^k}$ is a rational homology ball. Now, consider the long exact sequence of the pair $(\bar{W}_{p^k}, \bar{X}_{p^k})$. Since $H_1(\bar{X}_{p^k})$ is torsion, by duality and the universal coefficient theorem $H_2(\bar{X}_{p^k}) = 0$ and we have the following portion of the sequence.

$$0 \to H_2(\bar{W}_{p^k}) \to H_2(\bar{W}_{p^k}, \bar{X}_{p^k}) \to H_1(\bar{X}_{p^k}) \to H_1(\bar{W}_{p^k}) \to H_1(\bar{W}_{p^k}, \bar{X}_{p^k}) \to 0$$

Also by duality and the universal coefficient theorem we have $H_1(\bar{W}_{p^k}, \bar{X}_{p^k}) = H_2(\bar{W}_{p^k})$ and $H_2(\bar{W}_{p^k}, \bar{X}_{p^k}) = H_1(\bar{W}_{p^k})$. For any such exact sequence of finite abelian groups an easy algebraic argument shows that the product of the orders of the odd terms equals the product of the orders of the even terms. Hence, in the present case we have $|H_1(\bar{X}_{p^k})| = |H_1(\bar{W}_{p^k})|^2/|H_2(\bar{W}_{p^k})|^2$. This quotient must be an integer, and hence is a square.

For the final observation, note that $\mathrm{Ker}(H_1(\bar{X}_{p^k}) \to H_1(\bar{W}_{p^k}))$ is isomorphic to $H_2(\bar{W}_{p^k}, \bar{X}_{p^k})/H_2(\bar{W}_{p^k})$; this quotient has order $|H_1(\bar{W}_{p^k})|/|H_2(\bar{W}_{p^k})|$, the square root of the order computed above. Obviously, the kernel of the inclusion is invariant under the deck transformation, as the transformation extends over $\bar{W}_{P^k}$. $\square$

7.10.3. Linking form information. Using the branched cover of the 4-ball over the slice disk of a knot we can also obtain information regarding the linking form of the branched cover of S^3. With the previous notation we have:

THEOREM 7.10.5. The kernel of the inclusion $H_1(\bar{X}_{p^k}) \to H_1(\bar{W}_{p^k})$ is self annihilating with respect to the linking form.

Proof Given cycles x_1 and x_2 representing classes in the kernel, pick a 2-chain z_1 in $\bar{X}_{p^k}$ with boundary rx (where r is the order of $H_1(\bar{X}_{p^k})$) and pick chains w_1 and w_2 in $\bar{W}_{p^k}$ with boundaries x_1 and x_2 respectively. Observe that the *cycle* $z_1 - rw_1$ intersects the relative cycle w_2 0 times since $z_1 - rw_1$ represents torsion in homology. This intersection number is given by $z_1 \cdot x_2 - rw_1 \cdot w_2$. Clearly then the linking number, $z_1 \cdot x_2/r$ is an integer, and hence is trivial in $\mathbb{Q}/\mathbb{Z}$. $\square$

7.10.4. Doubly null concordant knots. Using $\bar{W}_{p^k}$ as above, we can derive invariants concerning the branched covers of S^3 over doubly null concordant knots. In terms of the homology the result in not quite as strong as the result obtained from the Seifert form, but we obtain immediate information regarding the linking form and also develop geometric methods that will be of use later.

THEOREM 7.10.6. If K is doubly null concordant then for all n, $H_1(\bar{X}_n) = \mathbb{Z}^k \oplus T \oplus T$ for some integer k and finite group T. Furthermore, the linking form of $\bar{X}_n$ vanishes on each T summand.

Proof Since the n-fold branched cover of S^4 branched over an unknotted 2-sphere is homeomorphic to S^4, we have that $\bar{X}_n$ embeds in S^4, splitting it into components, W_1 and W_2.

From the Mayer-Vietoris sequence, we have that $H_1(\bar{X}_n) = H_1(W_1) \oplus H_1(W_2)$. The following sequence of isomorphisms follows from the long exact sequence, excision (excising W_1) and duality.

$$H_1(W_1) = H_2(S^4, W_1) = H_2(W_2, \bar{X}_n) = H^2(W_2)$$

Hence, by the universal coefficient theorem applied to the last term, we have that the torsion subgroups of $H_1(W_1)$ and $H_1(W_2)$ are isomorphic. Since the Mayer-Vietoris map above is given by the inclusion, it is clear that that the two T summands correspond to the kernels of the inclusions of $\bar{X}_n$ into W_1 and W_2.

Now we want to show that two elements in the kernel of the inclusion $\bar{X}_n \to W_1$ link trivially. We proceed as before, using the same notation, but there is the complication that W_1 need not be a rational homology ball, since we are not assuming that n is a prime power. But notice that in the previous argument we only used that W_1 was a rational homology ball in showing that the *integral* intersection number $(z_1 - rw_1) \cdot w_2$ was trivial. In the present case we adjust the argument by noting that showing that this intersection is trivial is equivalent to showing that $(z_1 - rw_1) \cdot rw_2 = 0$. We can write $rw_2 = \partial(z_2)$ for a chain z_2 in $\bar{X}_n$, and then note that $(z_1 - rw_1) \cdot rw_2 = (z_1 - rw_1) \cdot (rw_2 - z_2)$, since the absolute class $(z_1 - rw_1) \cdot rw_2$ can be represented in the interior of W_1 and z_2 is in the boundary. Finally, we observe that the intersection of these two absolute classes is 0, since the intersection can be viewed as taking place in S^4 where all intersection numbers of 2-dimensional cycles are trivial. □

7.11. Witt class invariants associated to W_n

The most important Witt class invariants associated with a knot arise from the $\mathbb{Q}[t, t^{-1}]$ intersection form on $H_2(W_\infty)$. Working effectively with these depends on the duality theory of the next chapter. However, very strong Witt class invariants arise from finite covers, and here fairly simple arguments are available. These results also offer a lot of intuition regarding the more general results.

For any knot K, let F be a surface bounded by K in B^4. Fix a prime power p^k and let ω denote ω_{p^k}, a primitive p^k-root of unity. Let $\bar{W}$ be the p^k-fold branched cover of B^4 branched over F. Finally, let B denote the $\mathbb{Q}(\omega)$ valued intersection form defined in 7.8.1 of Section 7.8. (Recall that this form can be defined via the standard intersection form restricted to the ω eigenspace of the action of the deck transformations on $H_2(\bar{W}_{p^k}, \mathbb{Q}(\omega))$, denoted $H_2(\bar{W}_{p^k})_\omega)$. It can also be defined via the twisted intersection form on $H_2(\bar{W}_{p^k}) \otimes_{\mathbb{Q}[\mathbb{Z}/p^k]} \mathbb{Q}(\omega)$.)

We have seen that B is nonsingular (since $\bar{X}_{p^k}$ is a rational homology sphere). Hence it defines an element in the Witt group $W(\mathbb{Q}(\omega))$. For now we denote its Witt class by $[B]$. The main results are that for fixed p^k, $[B]$ depends only on K and not on the choice of surface F, and that $[B]$ is a concordance invariant.

THEOREM 7.11.1. With the notation of the preceding paragraph, $[B]$ is a knot invariant.

Proof Suppose F_1 and F_2 are two surfaces in B^4 bounded by K. Then the union $(B^4, F_1) \cup (B^4, F_2) = (S^4, G)$ yields a closed surface in the 4-sphere. Associated to each pair (B^4, F_i) we have an associated Witt class, $[B_i]$, and in the same way we have associated to (S^4, G) a Witt class, say $[C]$ via its p^k-fold branched cover.

Our first observation is that $[C] = [B_1] + [B_2]$. This follows from the fact that the p^k-fold branched cover of (S^4, G) is the union of the p^k-fold branched covers of (B^4, F_1) and (B^4, F_1) along a rational homology sphere.

To conclude the proof, we next need to show that $[C]$ is trivial. To see this, we first observe that $G = \partial M$ for some 3-manifold M embedded in S^4. (See the proof of Theorem 4.1.1.) Hence, (S^4, G) bounds (B^5, M^*), where M^* is a pushed in copy of M. It follows that the p^k-branched cover, $\bar{W}$, of (S^4, G) bounds the branched cover of p^k-branched cover, $\bar{Y}$, of (B^5, M^*). Finally, the form $[B]$ vanishes on the kernel of the inclusion $H_2(\bar{W}) \to H_2(\bar{Y})$, and this kernel is half dimensional. (To deduce this dimension result, consider the long exact sequence of the pair $(\bar{Y}, \bar{W})$ in conjunction with the fact that $H_1(\bar{W}, \mathbb{Q}(\omega)) = H_3(\bar{W}, \mathbb{Q}(\omega)) = 0$.) $\square$

THEOREM 7.11.2. Continuing with the previous notation, if K is slice, $[B] = 0$ in $W(\mathbb{Q}(\omega))$.

Proof If K is slice a slice disk for K can be used as the surface F in the definition of B. In this case $H_2(\bar{W}, \mathbb{Q}(\omega)) = 0$, so the associated Witt class is of course trivial as well. $\square$

Via the identification of the Witt form of the intersectin form of the branched cover with the Seifert matrix, we see that this Witt class invariant is the equivalent to the one defined in Chapter 3.

7.11.1. Applications to 4-ball genus and unknotting number. We conclude this section with that the observation the previous work yields bounds on the 4-ball genus of a knot, $g_4(K)$ and hence on its unknotting number, $u(K)$. These arise since the rank of the form B above is bounded by twice the genus of any surface bounded by K in the 4-ball. Hence:

THEOREM 7.11.3. If K bounds an embedded surface of genus g in the 4-ball, then the Tristram-Levine signatures of a knot, $\sigma_\omega(K)$, satisfy $\sigma_\omega(K) \leq 2g$.

EXAMPLE 7.11.4. If K is the trefoil knot, then the connected sum of n copies of K, $\#_n K$, has 4-ball genus and unknotting number n.

7.12. Duality and the Infinite Cyclic Cover

In the previous sections we saw the essential role played by Poincare duality in the analysis of the homology and intersection form of finite branched cyclic covers of S^3 and B^4. We will now briefly discuss two forms of duality that exist in the non-compact setting that arises in considering infinite cyclic covers.

First we will consider homology and cohomology groups with twisted coefficients, and Poincare duality for these groups. These results are sufficient for the analysis of the intersection form of the infinite cyclic cover of $B^4 - F$ and of the Blanchfield pairing of a knot. We then consider Milnor duality of infinite cyclic covers; this result states that under appropriate constraints, the homology and cohomology of the infinite cyclic cover of an n-dimensional manifold satisfy the duality properties of a manifold of dimension $n-1$. With this we arrive at a natural way of associating an isometric structure to a knot and are able to show that it offers a concordance invariant. There are geometric relations between the 4-dimensional intersection form, the Blanchfield pairing, and the isometric structure. We will see that these correspond to the maps constructed at the algebraic level in the previous chapters.

7.12.1. Twisted Coefficients for Homology and Cohomology. Let X be a compact manifold. In our examples X will be a knot exterior in S^3 or the exterior of a surface in B^4. We assume that some triangulation or CW structure on X is fixed. Next, let $\Pi = \pi_1(X)$ and let A be a left $\mathbb{Z}[\Pi]$ module. Then we define the twisted homology and cohomology of X with A coefficients as follows.

Denote by $\tilde{X}$ the universal cover of X and let $C_*(\tilde{X})$ be the simplicial (or CW) chain complex of that cover; $C_*(\tilde{X})$ is a right $\mathbb{Z}[\Pi]$ module. We can form the tensor product $C_*(\tilde{X}) \otimes_{\mathbb{Z}[\Pi]} A$. The homology of this complex is the twisted homology of X with A coefficients, denoted $H_*(X, A)$.

In defining the cohomology a little care must be taken with the modules, as it is better to work with right modules rather than left. We can form the chain complex $\mathrm{Hom}_{\mathbb{Z}[\Pi]}(C_*(\tilde{X}), \bar{A})$, where $\bar{A}$ is A with the right $\mathbb{Z}[\Pi]$ module structure given by $a\alpha = \alpha^{-1}a$. The homology of this chain complex gives the twisted cohomology of X with A coefficients, denoted $H^*(X, A)$.

Notice, if for some ring R, A has the structure of a right R module that is compatible with its left $\mathbb{Z}[\Pi]$ structure (eg. $(\alpha a)r = \alpha(ar)$) then the homology and cohomology groups of X with A coefficients form right R modules. In particular, this occurs in the case that $A = \mathbb{Z}[\Pi]$ with the natural Π action. In this case, observe that $H_*(X, \mathbb{Z}[\Pi]) = H_*(\tilde{X})$ with group action given by the action of the group of deck transformations.

Finally, one must prove that the resulting groups are independent of the choice of triangulation. This follows from a subdivision theorem or can be proved by switching to singular homology. We follow the simplicial approach for the geometric intuition it offers, along with a simple description of Poincare duality we outline presently.

EXAMPLE 7.12.1. For a simple example, consider the case of $X = S^1$. With the usual structure of X having a single 0-cell and a 1-cell, the chain complex of the universal cover is

$$0 \to \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}] \to 0.$$

If we let T denote the automorphism of A generating a $\mathbb{Z}$ action, then after tensoring this chain complex with A we arrive at

$$0 \to A \overset{T-1}{\to} A \to 0.$$

Hence, $H_0(X, A) = \mathrm{Coker}(T-1)$ and $H_1(X, A) = \mathrm{Ker}(T-1)$. As three specific cases we have:

a) if $A = \mathbb{Z}$ with the action of T given by $T(x) = x$, then $H_0(X, A) = \mathbb{Z}$ and $H_1(X, A) = \mathbb{Z}$.
b) if $A = \mathbb{Z}$ with the action of T given by $T(x) = -x$, then $H_0(X, A) = \mathbb{Z}_2$ and $H_1(X, A) = 0$.
c) if $A = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$ with the natural action, $T(p(t)) = tp(t)$, then $H_0(X, A) = \mathbb{Z}[t, t^{-1}]/ < t - 1 >$ and $H_1(X, A) = 0$. (Of course, this follows from the observation that $H_*(X, \mathbb{Z}[\mathbb{Z}]) = H_*(\tilde{X})$, and in this example $\tilde{X} = \mathbb{R}^1$.)

In the same situations we can compute the cohomology. Note that $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z}[\mathbb{Z}], \bar{A})$ is naturally isomorphic to A as an abelian group. With this it is easy to show that $H^0(X, A) = \mathrm{Ker}(T^{-1} - 1)$ and $H^1(X, A) = \mathrm{Coker}(T^{-1} - 1)$,

EXAMPLE 7.12.2. Consider the 3-dimensional lens space, $L(p, q)$ introduced in Exercise 2.6.12. It can be seen that it has the structure of a cell complex with one cell in each dimension less than 4. The associated chain complex on the universal cover (S^3 is the p-fold universal cover) is

$$0 \to \mathbb{Z}[\mathbb{Z}_p] \overset{t^q-1}{\to} \mathbb{Z}[\mathbb{Z}_p] \overset{N}{\to} \mathbb{Z}[\mathbb{Z}_p] \overset{t-1}{\to} \mathbb{Z}[\mathbb{Z}_p] \to 0,$$

where t is the generator of Π represented by the 1-cell, and N is the norm element, the sum $\sum_{i=0}^{p-1} t^i$.

As an example of a $\mathbb{Z}[\mathbb{Z}_p]$ module, consider $\mathbb{Z}[\omega_p]$, where ω_p is a primitive pth root of unity. In this case the norm element becomes trivial, and for homology we have the following sequence.

$$0 \to \mathbb{Z}[\omega_p] \stackrel{{\omega_p}^q-1}{\to} \mathbb{Z}[\omega_p] \stackrel{0}{\to} \mathbb{Z}[\omega_p] \stackrel{\omega_p-1}{\to} \mathbb{Z}[\omega_p] \to 0$$

Since the quotient $\mathbb{Z}[\omega_p]/ < 1-\omega_p >= \mathbb{Z}_p$, one concludes that the homology groups are given by $H_0(L(p,q),\mathbb{Z}[\omega_p]) = \mathbb{Z}_p$, $H_1(L(p,q),\mathbb{Z}[\omega_p]) = 0$, $H_2(L(p,q),\mathbb{Z}[\omega_p]) = \mathbb{Z}_p$, and $H_3(L(p,q),\mathbb{Z}[\omega_p]) = 0$. The result for cohomology is the same, only with the dimensions switched from n to $3-n$.

7.12.2. Infinite cyclic covers and $\mathbb{F}[t,t^{-1}]$ coefficients. We now make an important observation. Suppose that a normal subgroup $N \subset \Pi$ acts trivially on A. Then A can be viewed as a Π/N module. There is a cover of X, denoted $\tilde{X}_N$ corresponding to N with group of deck transformations Π/N. Hence we can form the chain complexes $C_*(\tilde{X}_N) \otimes_{\mathbb{Z}[\Pi/N]} A$ and $\mathrm{Hom}_{\mathbb{Z}[\Pi/N]}(C_*(\tilde{X}_N),\bar{A})$. The following theorem is immediate.

THEOREM 7.12.3. In the situation above, the twisted homology and cohomology groups, $H_*(X,A)$ and $H^*(X,A)$, are given by the homology groups of the complexes $C_*(\tilde{X}_N) \otimes_{\mathbb{Z}[\Pi/N]} A$ and $\mathrm{Hom}_{\mathbb{Z}[\Pi/N]}(C_*(\tilde{X}_N),\bar{A})$, respectively.

COROLLARY 7.12.4. In the situation described above, $H_*(X,\mathbb{Z}[\Pi/N]) = H_*(\tilde{X}_N)$.

Proof We just note that the complexes $C_*(\tilde{X}_N) \otimes_{\mathbb{Z}[\Pi/N]} \mathbb{Z}[\Pi/N]$ and $C_*(\tilde{X}_N)$ are isomorphic.

EXAMPLE 7.12.5. The corresponding statement for cohomology is not true; for instance, consider $H^1(S^1,\mathbb{Z}[t,t^{-1}])$ as in Example above. It is easy to see the source of the failure; in general, $\mathrm{Hom}_{\mathbb{Z}[\Pi/N]}(C_*(\tilde{X}_N),\mathbb{Z}[\Pi/N]) \neq \mathrm{Hom}_{\mathbb{Z}}(C_*(\tilde{X}_N),\mathbb{Z})$. For example, note that $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z}[\mathbb{Z}],\mathbb{Z}[\mathbb{Z}]) = \mathbb{Z}[\mathbb{Z}]$ but $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\mathbb{Z}],\mathbb{Z})$ is uncountable. Understanding the twisted cohomology is best done via the universal coefficient theorem, as we describe below.

EXAMPLE 7.12.6. We now come to one of the most important examples. Suppose that we are given a homomorphism of $\pi_1(X)$ to $\mathbb{Z}$. Then there is a natural action of $\pi(X)$ on $\mathbb{Z}[t,t^{-1}]$, and we can form groups $H_*(X,\mathbb{Z}[t,t^{-1}])$ and $H^*(X,\mathbb{Z}[t,t^{-1}])$. We can also form the associated infinite cyclic cover of X, X_∞, and by the previous theorem we have that $H_*(X,\mathbb{Z}[t,t^{-1}]) = H_*(X_\infty,\mathbb{Z})$.

7.12.3. Universal coefficient theorem. The universal coefficient theorem applies to relate the cohomology of homology of R chain complexes as long as R is a PID. Unfortunately, $\mathbb{Z}[\Pi]$ is not a PID, nor is $\mathbb{Z}[t, t^{-1}]$), the ring of perhaps the greatest interest to us. However, by switching from $\mathbb{Z}$ to a field $\mathbb{F}$ we have a PID, $\mathbb{F}[\mathbb{Z}] = \mathbb{F}[t, t^{-1}]$. In this case the universal coefficient theorem applies to give the following.

THEOREM 7.12.7. There is a short exact sequence

$$0 \to \mathrm{Ext}_{\mathbb{F}[\mathbb{Z}]}(H_{i-1}(X, \mathbb{F}[\mathbb{Z}]), \overline{\mathbb{F}[\mathbb{Z}]}) \to H^i(X, \mathbb{F}[\mathbb{Z}]) \to \mathrm{Hom}_{\mathbb{F}[\mathbb{Z}]}(H_i(X, \mathbb{F}[\mathbb{Z}]), \overline{\mathbb{F}[\mathbb{Z}]}) \to 0.$$

LEMMA 7.12.8. To apply these result we need the following calculations for $p(t)$ a nontrivial polynomial.

$\mathrm{Ext}_{\mathbb{F}[\mathbb{Z}]}(\mathbb{F}[\mathbb{Z}], \overline{\mathbb{F}[\mathbb{Z}]}) = 0$
$\mathrm{Hom}_{\mathbb{F}[\mathbb{Z}]}(\mathbb{F}[\mathbb{Z}], \overline{\mathbb{F}[\mathbb{Z}]}) = \mathbb{F}[\mathbb{Z}]$
$\mathrm{Ext}_{\mathbb{F}[\mathbb{Z}]}(\mathbb{F}[\mathbb{Z}]/ < p(t) >, \overline{\mathbb{F}[\mathbb{Z}]}) = \mathbb{F}[\mathbb{Z}]/ < p(t^{-1}) >$
$\mathrm{Hom}_{\mathbb{F}[\mathbb{Z}]}(\mathbb{F}[\mathbb{Z}]/ < p(t) >, \overline{\mathbb{F}[\mathbb{Z}]}) = 0.$

7.12.4. Poincare Duality for Twisted Coefficients. In short, if M is a closed oriented n-manifold then one has an isomorphism, referred to as the Poincare duality isomorphism, $PD : H_r(M, A) \to H^{n-r}(M, A)$. If A is a $\mathbb{Z}[\Pi]$ bimodule (that is, if it has a compatible right $\mathbb{Z}[\Pi]$ structure) then PD is a homomorphism of right $\mathbb{Z}[\Pi]$ modules. We will not present the details of the proof but want to give enough of a description to motivate the applications that follow.

For the given triangulation T of M, there is a dual CW structure, T^*, constructed as follows. For each r-simplex σ of T, there is an $n-r$ cell, σ^*, of T^*, formed as the closed link of σ in the first barycentric subdivision of T. Notice that σ and σ^* meet in exactly one point, the barycenter of σ.

This construction lifts to the the universal cover of M, $\tilde{M}$ giving a triangulation and dual cell structure, $\tilde{T}$ and $\tilde{T}^*$. Using these decompositions we set up the following correspondence. For each r-simplex σ of $\tilde{T}$ there is an element of $\mathrm{Hom}_{\mathbb{Z}[\Pi]}(\tilde{T}^*_{n-r}, \overline{\mathbb{Z}[\Pi]})$ defined by $\sigma(\eta) = \sum_{g \in G} (\sigma g \cap \eta) g^{-1}$, where $(\sigma g \cap \eta)$ is just the integer intersection number of the g translate of σ and η.

One can now check that this correspondence determines a chain isomorphism from $C_*(\tilde{M}) \otimes_{\mathbb{Z}[\Pi]} \mathbb{Z}[\Pi]$ to $\mathrm{Hom}_{\mathbb{Z}[\Pi]}(C_*(\tilde{M}), \overline{\mathbb{Z}[\Pi]})$ carrying elements of degree r to elements of degree $n-r$. Hence, we have that $H_r(M, \mathbb{Z}[\Pi])$ is isomorphic to $H^{n-r}(M, \mathbb{Z}[\Pi])$. Finally, to achieve duality with arbitrary coefficients, we notice that since this is a chain isomorphism, it remains so after tensoring with a $\mathbb{Z}[\Pi]$ module A. Since $\mathrm{Hom}_{\mathbb{Z}[\Pi]}(C_*(\tilde{M}), \overline{\mathbb{Z}[\Pi]}) \otimes_{\mathbb{Z}[\Pi]} A$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}[\Pi]}(C_*(\tilde{M}), \bar{A})$, we get the desired general theorem.

In the case that M is a compact manifold with boundary one has a similar duality theorem, only now the pairing is between the absolute and relative homology or cohomology groups, as in the classical setting. Also, the entire discussion applies with $\mathbb{Z}$ replaced by a field.

7.12.5. The homology of the infinite cyclic cover of 0-surgery on S^3 along a knot. As our first application, we consider the closed manifold obtained by 0 surgery on K, denoted $M(K,0)$ and denote its infinite cyclic cover by $\bar{X}_\infty$. This is closely related to the infinite cyclic cover of the knot complement, $\tilde{X}_\infty$, considered in Section 7.1. In fact their homologies are identical.

We have that $H_i(\bar{X}_\infty, \mathbb{Q}) = H_i(M(K,0), \mathbb{Q}[t,t^{-1}])$ as $\mathbb{Q}[t,t^{-1}]$ modules. Therefore:

a. $H_0(M(K,0), \mathbb{Q}[t,t^{-1}]) = \mathbb{Q}[t,t^{-1}]/ < t-1 >$
b. $H_1(M(K,0), \mathbb{Q}[t,t^{-1}]) = \oplus_i \mathbb{Q}[t,t^{-1}]/ < p_i^1(t) >$
c. $H_2(M(K,0), \mathbb{Q}[t,t^{-1}]) = \mathbb{Q}[t,t^{-1}]/ < t-1 > \oplus_i \mathbb{Q}[t,t^{-1}]/ < p_i^2(t) >$
d. $H_3(M(K,0), \mathbb{Q}[t,t^{-1}]) = 0$

where in each case there are a finite number of summands and the $p_i^j(t)$ are prime to $t-1$.

Applying duality and the universal coefficient theorem implies that there is only one summand in the H_2 term: $H_2(M(K,0), \mathbb{Q}[t,t^{-1}]) = \mathbb{Q}[t,t^{-1}]/ < t-1 >$. Poincare duality has an implication for H_1 as well; for instance, it gives the symmetry of the Alexander polynomial: $\Delta_K(t) = \prod_i p_i^1(t) = \prod_i p_i^1(t^{-1})$.

7.12.6. The homology of the infinite cyclic cover of $B^4 - F$. As in Section 7.8, let (B^4, F) be a pair with F an oriented properly embedded, connected, genus g surface in the 4-ball with $\partial(B^4, F) = (S^3, K)$. Let W denote $B^4 - N(F)$. The boundary of W, ∂W is constructed from the complement of K by attaching $S^1 \times F$ along the boundary (so that if $g = 0, \partial W = M(K,0)$; see 3.2.1). Concerning the boundary we have :

a. $H_0(\partial W, \mathbb{Q}[t,t^{-1}]) = \mathbb{Q}[t,t^{-1}]/ < t-1 >$.
b. $H_1(\partial W, \mathbb{Q}[t,t^{-1}]) = (\mathbb{Q}[t,t^{-1}]/ < t-1 >)^{2g} \oplus_i \mathbb{Q}[t,t^{-1}]/ < p_i^1(t) >$.
c. $H_2(\partial W, \mathbb{Q}[t,t^{-1}]) = \mathbb{Q}[t,t^{-1}]/ < t-1 > \oplus_i \mathbb{Q}[t,t^{-1}]/ < p_i^2(t) >$.
d. $H_3(\partial W, \mathbb{Q}[t,t^{-1}]) = 0$.

From the results of Section 7.8, in particular Lemma 7.8.1 we have:

a. $H_0(W, \mathbb{Q}[t,t^{-1}]) = \mathbb{Q}[t,t^{-1}]/ < t-1 >$

b. $H_1(W, \mathbb{Q}[t, t^{-1}]) = \oplus_i \mathbb{Q}[t, t^{-1}]/ < q_i^1(t) >$
c. $H_2(W, \mathbb{Q}[t, t^{-1}]) = (\mathbb{Q}[t, t^{-1}])^{2g} \oplus_i \mathbb{Q}[t, t^{-1}]/ < q_i^2(t) >$
d. $H_3(W, \mathbb{Q}[t, t^{-1}]) = \oplus_i \mathbb{Q}[t, t^{-1}]/ < q_i^3(t) >$
e. $H_4(W, \mathbb{Q}[t, t^{-1}]) = 0$

where each direct sum is finite and each q_i^j is relatively prime to $t^{p^k} - 1$ for all prime p and nonnegative k.

Applying Poincare duality, the universal coefficient theorem and the long exact sequence we make the following series of observations:

a. $H_0(W, \partial W; \mathbb{Q}[t, t^{-1}]) = 0$ and $H_4(W, \partial W; \mathbb{Q}[t, t^{-1}]) = 0$.
b. Since $H_0(W, \partial W; \mathbb{Q}[t, t^{-1}]) = 0$, it follows that $H_3(W, \mathbb{Q}[t, t^{-1}])$ is torsion free, and hence $H_3(W, \mathbb{Q}[t, t^{-1}]) = 0$.
c. $H_3(W, \partial W; \mathbb{Q}[t, t^{-1}]) = \mathbb{Q}[t, t^{-1}]/ < t-1 >$ and the map $H_3(W, \partial W; \mathbb{Q}[t, t^{-1}]) \to H_3(\partial W; \mathbb{Q}[t, t^{-1}])$ is an isomorphism.
d. $H_1(W, \partial W; \mathbb{Q}[t, t^{-1}]) = \oplus_i \mathbb{Q}[t, t^{-1}]/ < q_i^2(t^{-1}) >$.
e. $H_2(W, \partial W; \mathbb{Q}[t, t^{-1}]) = (\mathbb{Q}[t, t^{-1}])^{2g} \oplus_i \mathbb{Q}[t, t^{-1}]/ < q_i^1(t^{-1}) >$.

The following result, which previously appeared as Exercise 4.3.2 to be proved using Seifert matrices, is a direct consequence:

THEOREM 7.12.9. If K is slice then $\Delta_K(t)$ factors as $f(t)f(t^{-1})$ for some polynomial f (up to multiplication by units in $\mathbb{Q}[t, t^{-1}]$).

Proof Let W be the complement of a slice disk for K. Since in this case $g = 0$ we have the exact sequence of $\Lambda = \mathbb{Q}[t, t^{-1}]$ torsion modules:

$$0 \to H_2(W; \Lambda) \to H_2(W, \partial W; \Lambda) \to H_1(\partial W; \Lambda) \to H_1(W; \Lambda) \to H_1(W, \partial W; \Lambda) \to 0.$$

Let $Q_1 = \prod q_i^1$ and let $Q_2 = \prod q_i^2$ in the previous notation. Then as Λ modules $H_1(W; \Lambda)$ has order $Q_1(t)$, $H_2(W; \Lambda)$ has order $Q_2(t)$, $H_1(W, \partial W; \Lambda)$ has order $Q_2(t^{-1})$, $H_2(W, \partial W; \Lambda)$ has order $Q_1(t^{-1})$ and $H_1(\partial W; \Lambda)$ has order $\Delta_K(t) = \prod p_i(t)$. The alternating products of these orders must be 1, so we have

$$\Delta_K(t) = Q_1(t)Q_1(t^{-1})/Q_2(t)Q_2(t^{-1}).$$

Since Δ is a polynomial, cancelation must occur, and the result follows. $\square$

7.12.7. The Blanchfield Pairing. There is an algebraic interpretation of the Blanchfield pairing defined geometrically in Section 7.7. We have the following sequence of isomorphisms:

$$H_1(M(K,0),\mathbb{Q}[t,t^{-1}]) \to H^2(M(K,0),\mathbb{Q}[t,t^{-1}]) \to \mathrm{Ext}(H_1(M(K,0),\mathbb{Q}[t,t^{-1}]),\overline{\mathbb{Q}[t,t^{-1}]})$$
$$\to \mathrm{Hom}(H_1(M(K,0),\mathbb{Q}[t,t^{-1}]),\overline{\mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}]}).$$

The second isomorphism is given by the universal coeffiecient theorem since $H_1(M(K,0),\mathbb{Q}[t,t^{-1}])$ is $\mathbb{Q}[t,t^{-1}]$ torsion. The last isomorphism follows from the long exact sequence associated to the short exact sequence of coefficients:

$$0 \to \mathbb{Q}[t,t^{-1}] \to \mathbb{Q}(t) \to \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}] \to 0.$$

Hence, we arrive at a nonsingular pairing

$$H_1(M(K,0),\mathbb{Q}[t,t^{-1}]) \times H_1(M(K,0),\mathbb{Q}[t,t^{-1}]) \to \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}].$$

It is an exercise to trace through the various maps and verify that this pairing is the same as the Blanchfield pairing defined previously. The details necessarily call on the proof of the universal coefficient theorem along with the long exact sequence associated to a sequence of coefficients. We now present some of the key ideas.

Let C_* be a free chain complex over a PID R, and, as usual, let $\mathbb{Z}_i$ and B_* denote the cycles and boundaries. There is a short exact sequence $0 \to B_i \to \mathbb{Z}_i \to H_i(C_*) \to 0$. From this we arrive at a surjection $f_1 : \mathrm{Hom}(B_i,R) \to \mathrm{Ext}(H_i,R)$. In fact, $\mathrm{Ext}(H_i,R) = \mathrm{Hom}(B_i,R)/\mathrm{Im}(\mathrm{Hom}(\mathbb{Z}_i,R))$ (this is almost the definition of Ext). We also have that $\mathrm{Hom}(B_i,R) = \mathrm{Hom}(C_{i+1}/\mathbb{Z}_{i+1},R)$, and this last group maps to $\mathrm{Hom}(C_{i+1},R)$, giving $f_2 : \mathrm{Hom}(B_i,R) \to \mathrm{Hom}(C_{i+1},R)$. Finally, there is a map, f_3 of the cocylces in $\mathrm{Hom}(C_{i+1},R)$ to $H_{i+1}(C_*)$. Assembling these maps appropriately gives the map $\mathrm{Ext}(H_i(C_*),R) \to H^{i+1}(C_*)$. To be a little more precise, the map is given by $f_3 f_2 {f_1}^{-1}$, which is well defined although ${f_1}^{-1}$ is not.

As the second crucial step, we observe that if $H_i(C_*)$ is R-torsion, there is a natural map from $\mathrm{Hom}(B_i,R)/\mathrm{Im}(\mathrm{Hom}(\mathbb{Z}_i,R))$ to $\mathrm{Hom}(H_i(C_*),QR/R)$ where QR is the field of quotients of R.

7.13. Milnor Duality

In [**?**] Milnor proved that the infinite cyclic cover of a knot complement has the homological and duality properties of a surface with connected boundary. Here we will present this result, first by outlining Milnor's homological argument and then giving a geometric description of the associated maps. It is through this duality that one finds a geometric source for isometric structures associated to concordance classes.

7.13.1. Duality and an Isometric Structure. Let M be a closed oriented n-dimensional manifold and let M_∞ be a connected infinite cyclic cover of M. Milnor's theorem in [**?**] states the following:

THEOREM 7.13.1. If $H_*(M_\infty, \mathbb{F})$ is finitely generated for a field $\mathbb{F}$, then $H^{n-1}(M_\infty, \mathbb{F})$ is one dimensional over $\mathbb{F}$, and the vector spaces $H^i(M_\infty, \mathbb{F})$ and $H^{n-1-i}(M_\infty, \mathbb{F})$ are dual to each other, being othrogonally paired to $H^{n-1}(M_\infty, \mathbb{F}) \cong \mathbb{F}$ by the cup product pairing.

Proof (outline) One begins with a compact submanifold C of M_∞ with the property that the translates of C cover M_∞. Define $N_p = t^pC \cup t^{p+1}C \cdots$, and N'_p similarly using the negative translates. It follows readily from the finite generation condition that for some postive s, the map $H_*(M_\infty, N_{p+s}, \mathbb{F}) \to H^*(M_\infty, N_p, \mathbb{F})$ is the zero map. Via duality, $H^*(M_\infty, N_p, \mathbb{F}) \to H^*(M_\infty, N_{p+s}, \mathbb{F})$ is also zero.

Next, from the Mayer-Vietoris sequence, one finds that $H^{i-1}(M_\infty, \mathbb{F}) \to H^i(M_\infty, N_p \cup N'_p, \mathbb{F})$ induces an isomorphism in the limit. Also, the limit gives the cohomology group with compact support, $H^i_{cpct}(M_\infty, \mathbb{F})$.

By standard Poincaré duality, $H^i_{cpct}(M_\infty, \mathbb{F}) \cong H_{n-i}(M_\infty, \mathbb{F})$. Hence, we have that $H^{i-1}(M_\infty, \mathbb{F}) \cong H_{n-i}(M_\infty, \mathbb{F})$. This gives that $H^{n-1}(M_\infty, \mathbb{F}) \cong \mathbb{F}$. The desired pairing is induced by the Poincaré duality pairing, $H^i_{cmpct}(M_\infty, \mathbb{F}) \otimes H_{n-i}(M_\infty, \mathbb{F}) \to \mathbb{F}$. □

With this pairing it is easy to define an isometric pairing associated to a knot. In the above discussion, let M denote $M(K, 0)$, 0-surgery on K. Then M_∞ corresponds to $\bar{X}_\infty$ of our earlier discussion. Hence we have a nonsingular skew symmetric cup product pairing on $W : H^1(\bar{X}_\infty, \mathbb{F}) \otimes H^1(\bar{X}_\infty, \mathbb{F}) \to \mathbb{F}$. The triple consisting of the vector space $H = H^1(X_\infty, \mathbb{F})$, this cup product pairing, and the deck transformation, t, together define an isometric structure.

One can prove in a standard way that if K is slice then W vanishes on a t-invariant half dimensional summand of H. (Such a summand arises as the image of the cohomology of the infinite cyclic cover of the complement of a slice disk for K in the 4-ball.)

We also note that a careful examination of the homological interpretation of the Blanchfield pairing shows that this isometric structure is the one that is induced on the level of Witt groups by the Blanchfield pairing.

7.13.2. The Geometry of Milnor Duality. The definition of the map χ from linking forms, $W(Q(t))$, to isometric structures, $W_I(Q(t))$, included an unexpect construction of Laurent expansions of rational functions. A look at the geometry of Milnor duality offers some intuition concerning this map.

Let $H_*^{inf}(\bar{X}, \mathbb{F})$ denote the homology of the infinite cyclic cover of 0-surgery on a knot, based on the chain complex of locally finite infinite chains. We want to observe that $H_2^{inf}(\bar{X}, \mathbb{F}) \cong H_1(\bar{X}, \mathbb{F})$. To define the map, we start with the lift of a capped off Seifert surface for K, say G. Intersections with G define a map $H_2^{inf}(\bar{X}, \mathbb{F}) \to H_1(\bar{X}, \mathbb{F})$. It is easily seen to be independent of the choice of G.

CHAPTER 8

Casson-Gordon invariants

Returning to the knot concordance group $\mathcal{C}$, we have already seen that

$$\mathcal{C} \stackrel{j}{\to} \mathcal{C}_{top} \stackrel{k}{\to} \mathcal{G} \cong (\oplus_1^\infty \mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/2\mathbb{Z}) \oplus (\oplus_1^\infty \mathbb{Z}/4\mathbb{Z}).$$

The homorphisms j and k are both onto. A natural question to ask next, is about the kernels of these homomorphisms, especially in light of the fact that in the analogous situation of higher odd-dimensional knots, the kernels are trivial making the concordance group $\mathcal{C}$ isomorphic to the countable direct sum on the right. In 1975 Casson and Gordon [**?**, **?**] proved that the kernel of the homomorphism $i = k \circ j$ from $\mathcal{C}$ to the algebraic concordance group $\mathcal{G}$ which sends the concordance class of a knot to the concordance class of its Seifert matrix is nontrivial, by constructing nontrivial elements in the kernel, and Jiang [**?**] expanded on this to show that $\mathrm{Ker}(i)$ contains a subgroup isomorphic to $\oplus_1^\infty \mathbb{Z}$. Along these lines it was shown in [**?**] that it also contains a subgroup isomorphic to $\oplus_1^\infty \mathbb{Z}/2\mathbb{Z}$. The 1980s saw two significant developments in the study of concordance. The first was based on Freedman's work [**?**, **?**] studying the structure of topological 4–manifolds. One consequence was that methods of Levine and those of Casson–Gordon apply in the topological locally flat category, rather than only in the smooth setting. It follows that the above mentioned results regarding the subgroup $\oplus_1^\infty \mathbb{Z} \oplus \oplus_1^\infty \mathbb{Z}/2\mathbb{Z}$ apply to the kernel of the map k from $\mathcal{C}_{top}$ to $\mathcal{G}$. More significant, Freedman proved that all knots with trivial Alexander polynomial are in fact slice in the topological locally flat category.

The other important development concerns the application of differential geometric techniques to the study of smooth 4–manifolds, beginning with the work of Donaldson [**?**, **?**] and including the introduction of Seiberg–Witten invariants and their application to symplectic manifolds, the use of the Thurston–Bennequin invariant [**?**, **?**], and recent work of Ozsváth and Szabó [**?**]. This work quickly led to the construction of smooth knots of Alexander polynomial one that are not smoothly slice, thereby showing $\mathrm{Ker}(j)$ to be nontrivial. It also facilitated a much deeper understanding of related issues, such as the 4–ball genus of knots. References are too numerous to enumerate here; a few will be included as applications are mentioned. To name one, using knot Floer homology, Jennifer Hom [?] recently proved the existence of an infinite rank direct summand of $\mathrm{Ker}(j)$, the subgroup of $\mathcal{C}$ generated by topologically slice knots.

Work of Cochran, Orr and Teichner, [?, ?], has revealed a deeper structure to the knot concordance group. In that work a filtration of $\mathcal{C}$ is defined:

$$\cdots \mathcal{F}_{2.0} \subset \mathcal{F}_{1.5} \subset \mathcal{F}_1 \subset \mathcal{F}_{.5} \subset \mathcal{F}_0 \subset \mathcal{C}.$$

It is shown that $\mathcal{F}_0$ corresponds to knots with trivial Arf invariant, $\mathcal{F}_{.5}$ corresponds to knots in the kernel of ϕ and all knots in $\mathcal{F}_{1.5}$ have vanishing Casson–Gordon invariants. Using von Neumann η–invariants, it has been proved in [?] that each quotient is infinite. This work places Levine's obstructions and those of Casson–Gordon in the context of an infinite sequence of obstructions, all of which reveal a finer structure to $\mathcal{C}$.

8.1. The work of Casson and Gordon

The "half lives, half dies" principle of Theorem 2.6.1, when applied to the Seifert form of a slice knot as in Chapter 4, yielded the theory of algebraic concordance. In Chapter 7, in the context of covering spaces we revisited it in 7.10.4. In the case that K is algebraically slice, Casson-Gordon invariants use covering spaces, and call on to 2.6.1, to offer a further obstruction to a knot being slice. We follow the basic description of [?].

Before we state the definition of the Cason–Gordon invariants, let's recall the linking form on $\mathrm{Torsion}(H_1(M))$, as defined in 7.6, for an oriented 3–manifold M. If x and y are curves representing torsion in the first homology, then $\mathrm{lk}(x, y)$ is defined to be $(c \cap y)/n \in \mathbb{Q}/\mathbb{Z}$, where c is a 2-chain with boundary nx. Intersections are defined via transverse intersections of chains, and the value of the linking form is independent of the many choices in its definition. For a closed oriented 3-manifold the linking form is nonsingular in the sense that it induces an isomorphism from $\mathrm{torsion}(H_1(M))$ to $\mathrm{Hom}(\mathrm{torsion}(H_1(M)), \mathbb{Q}/\mathbb{Z})$.

Such a symmetric pairing on a finite abelian group, $l\colon\ H \times H \to \mathbb{Q}/\mathbb{Z}$, is called *metabolic* with *metabolizer* L if the linking form vanishes on $L \times L$ for some subgroup L with $|L|^2 = |H|$.

Now let K be a knot and let M_q be its q-fold branched cover, where q is the power of some prime number. We will call the corresponding cover of the 0-surgery, $\overline{M}_q$. As remarked in 7.5.3, $H_1(\overline{M}_q) = H_1(M_q) \oplus \mathbb{Z}$

Let x be an element of self-linking 0 in $H_1(M_q)$ and suppose that x is of prime power order, say p. Linking with x defines a homomorphism $\chi_x\colon\ H_1(M_q) \to \mathbb{Z}_{/}p\mathbb{Z}$. Furthermore,

χ_x extends to give a $\mathbb{Z}/p\mathbb{Z}$–valued character on $H_1(\overline{M}_q)$ which vanishes on the meridian of $\tilde{K}$. In turn, this character extends to give $\overline{\chi}_x\colon H_1(\overline{M}_q) \to \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}$. Since x has self-linking 0, bordism theory implies that the pair $(\overline{M}_q, \overline{\chi}_x)$ bounds a 4–manifold and character pair (W, η).

More generally, for any character $\chi\colon H_1(M_q) \to \mathbb{Z}_p$, there is a corresponding character $\overline{\chi}\colon H_1(\overline{M}_q) \to \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}$. This character might not extend to a 4–manifold, but since the relevant bordism groups are finite, for some multiple $r\overline{M}_q$ the character given by $\overline{\chi}$ on each component does extend to a 4–manifold, character pair, (W, η).

Let Y denote the $\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}$ cover of W corresponding to η. Using the action of $\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}$ on $H_2(Y, \mathbb{C})$ one can form the twisted homology group $H_2^t(W, \mathbb{C}) = H_2(Y, \mathbb{C}) \otimes_{\mathbb{C}[\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}]} \mathbb{C}(t)$. (The action of $\mathbb{Z}/p\mathbb{Z}$ on $\mathbb{C}(t)$ is given by multiplication by $e^{2\pi i/p}$.) There is a non-singular hermitian form on $H_2^t(W, \mathbb{C})$ taking values in $\mathbb{C}(t)$. The Casson-Gordon invariant is defined to be the difference of this form and the intersection form of $H_2(W, \mathbb{C})$, both tensored with $\frac{1}{r}$, in $W(\mathbb{C}[t, t^{-1}]) \otimes \mathbb{Q}$. (In showing that this Witt class yields a well-defined obstruction to slicing a knot, the fact that $\Omega_4(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z})$ is nonzero appears, and as a consequence one must tensor with $\mathbb{Q}$ to arrive at a well defined invariant, even in the case of χ_x in which it is possible to take $r = 1$.)

Definition 8.1.1. The Casson-Gordon invariant $\tau(M_q, \chi)$ is the class $(H_2^t(W, \mathbb{C}) - H_2(W), \mathbb{C})) \otimes \frac{1}{r} \in W(\mathbb{C}(t)) \otimes \mathbb{Q}$. We define $\sigma(K, \chi) = \sigma_1 \tau(K, \chi)$, as the signature obtained by evaluating a representative of the class at a unit complex number and taking the limit of the signature of the resulting complex valued form as the unit complex number approaches one.

The signature defined above induces a homomorphism $\sigma\colon W(\mathbb{C}(t)) \otimes \mathbb{Q} \to \mathbb{Q}$.

8.1.1. Genus One Knots and the Seifert Form. Gilmer observed in [**?, ?**] that for genus one knots the computation of Casson-Gordon invariants is greatly simplified. Roughly, he interpreted the Casson-Gordon signature invariants of an algebraically slice genus one knot in terms of the signatures of knots tied in the bands of the Seifert surface. The previous example offers an illustration of the appearance of these signatures. This work is now most easily understood via the use of companionship just described.

In short, if an algebraically slice knot K bounds a genus one Seifert surface F, then some nontrivial primitive class in $H_1(F)$ has trivial self-linking with respect to the Seifert form. If that class is represented by a curve α, the surface can be deformed to be a disk with two bands attached, one of which is tied into the knot α. If a new knot is formed by adding the knot $-\alpha$ to the band, the knot becomes slice and certain of its Casson-Gordon invariants will vanish. However, the previous results on companionship determine how the modification of the knot changes the Casson-Gordon invariant. The situation is made somewhat more delicate in that α is not unique: for genus one algebraically slice knots there are two metabolizers. The following represents the sort of result that can be proved.

Theorem 8.1.2. Let K be a genus one slice knot. The Alexander polynomial of K is given $(at-(a+1))((a+1)t-a)$ for some a. For some simple closed curve α representing a generator of a metabolizer of the Seifert form and for some infinite set of primes powers q, one has

$$\sum_{i=1}^{q} \sigma_{bm^i/p}(\alpha) = 0$$

for all prime power divisors p of $(a-1)^q - a^q$, and for all integers b.

(The appearance of the term $(a-1)^q - a^q$ represents the square root of the order of the homology of the q–fold branched cover.) Since the sum is taken over a coset of the multiplicative subgroup of $\mathbb{Z}/p\mathbb{Z}$, by combining these cosets one has the following.

Corollary 8.1.3. If K is a genus one slice knot with nontrivial Alexander polynomial, then for some simple closed curve α representing a generator of a metabolizer of the Seifert form, there is an infinite set of prime powers p for which

$$\sum_{i=1}^{p-1} \sigma_{i/p}(\alpha) = 0.$$

A theorem of Cooper [?] follows quickly:

Corollary 8.1.4. If K is a genus one slice knot with nontrivial Alexander polynomial, then for some simple closed curve x representing a generator of a metabolizer of the Seifert form,

$$\int_0^{1/2} \sigma_t(x)dt = 0.$$

(This theorem reappears in [?] where the integral is reinterpreted as a metabelian von Neumann signature of the original knot K, giving a direct reason why it is a concordance invariant. For more on this, see Section 10.3.)

Example 8.1.5. Note that in $K(a,b,c)$ of Figure 8.1,above, the numbers a, b indicate full twists in bands, whereas c indicates how many time the right band is going over

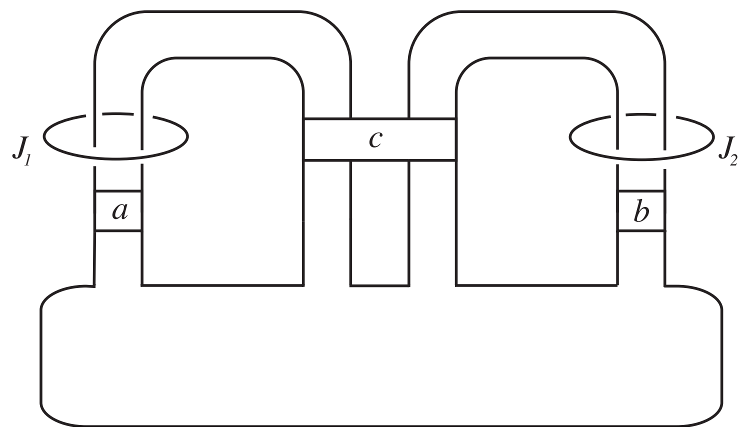

FIGURE 8.1. Pretzel knot $K(a, b, c)$ with linking curves J_1, J_2

the left. Consider the knot $K(0, 0, 3)$. Replacing the curves labeled J_1 and J_2 with the complements of knots J_1 and J_2 yields a knot for which the metabolizers of the Seifert form are represented by the knots J_1 and J_2. The knot is algebraically slice, but by the previous corollary, if both of the knots have signature functions with nontrivial integral, the knot is not slice.

EXERCISE 8.1.6. See another description of pretzel knots in Figure 2.25 of Example 2.8.2. How are a, b above related to p, q, r in Example 2.8.2?

8.2. The Casson-Gordon theorem

Let K be a knot in S^3, let p and q be powers of some primes, let M be a q-fold cyclic cover of S^3 branched over K, and let $\chi : H_1(M) \to \mathbb{Q}/\mathbb{Z}$ be an order p character. Let $\sigma(K, \chi)$ be the Casson-Gordon signature invariant as in Definition 8.1.1.

THEOREM 8.2.1. If K is slice, then there is a subgroup (metabolizer) H of $H_1(M)$, such that

(1) $H_1(M)/H \cong H$; consequently, order $(H) = \sqrt{\text{order} H_1(M)}$,
(2) the linking form on $H_1(M)$ vanishes on H, and
(3) the Casson-Gordon invariant $\sigma(K, \chi)$ is 0 for characters that vanish on H.

Proof

If K is slice, the 0-surgery on S^3 along K is obtained by removing a tubular neighborhood of the slice disk from the 4-ball (see 3.2.1), and the manifold W in Definition

8.1.1 can be taken to be the q-fold cover of this slice disk exterior. The metabolizer H corresponds to the kernel of the map induced by inclusion of ∂W into W. (Compare 2.6.1.) The characters that vanish on H are the ones that extend to $H_1(W)$. The result is a consequence of properties of the 4-manifold W obtained in this manner, with one caution.

If the linking form is nonsingular, this implies it has signature 0. However, since $H_2(\partial W) \neq 0$, the intersection from on $H_2(W)$ might be singular. To fix this, as described just prior to Definition 8.1.1, let $\partial(W, \eta) = r(\overline{M}, \overline{\chi})$, let Y be the $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}$ cover of W corresponding to η, and consider the manifold Y' built from Y by adding $B^3 \times B^2$ via a map of $B^3 \times S^1$ to W with the core S^1 mapping to a generator of $H_1(W)$. The boundary of Y' is now a homology sphere and the intersection form is now nonsingular. However, the operation of adding $B^3 \times B^2$ did not change H_2, so the original form is nontrivial. The argument works as long as that cover has homology $\mathbb{Z}\oplus$ torsion. This is the case at hand since we are working with prime powers. □

Next, we state Gilmer's additivity theorem [**?**, **?**] and a vanishing result proved by Litherland [**?**, Corollary B2].

THEOREM 8.2.2. Given a knot $K = K_1 \# K_2$, we have $M_K = M_{K_1} \# M_{K_2}$ and any order d character χ on $H_1(M_K)$ can be written as $\chi_1 \oplus \chi_2$. In this case we have $\sigma(K_1 \# K_2, \chi_1 \oplus \chi_2) = \sigma(K_1, \chi_1) + \sigma(K_2, \chi_2)$.

THEOREM 8.2.3. If χ is the trivial character, then $\sigma(K, \chi) = 0$.

The following is an easy consequence of Definition 8.1.1.

THEOREM 8.2.4. For every character χ, $\sigma(K, \chi) = \sigma(K, -\chi)$.

8.3. Intersection forms of 4–manifold

For a 4–manifold M we have the symmetric intersection pairing $H_2(M) \times H_2(M) \to \mathbb{Z}$. This can be defined as the algebraic intersection number of immersed surfaces representing 2–dimensional homology classes. More directly, via duality we have the pairing

$$H^2(M, \partial M) \times H^2(M, \partial M) \to H^2(M, \partial M) \times H^2(M) \to \mathbb{Z}$$

given by $(x, y) \to (x \cup r(y)) \cap [M, \partial M]$, where r is the restriction map from $H^2(M, \partial M)$ to $H^2(M)$.

Working with field coefficients, the pairing $H^2(M, \partial M, \mathbb{F}) \times H^2(M, \mathbb{F}) \to \mathbb{F}$ is nonsingular. If ∂M is a rational homology sphere, then $r\colon H^2(M, \partial M, \mathbb{Q}) \to H^2(M, \mathbb{Q})$ is an isomorphism, and the rational intersection form $H_2(M, \mathbb{Q}) \times H_2(M, \mathbb{Q}) \to \mathbb{Q}$ is nonsingular. Also, if $\partial M = \phi$ then the rational pairing is nonsingular.

In all cases, we define the signature of M to be the signature of the intersection form $H_2(M, \mathbb{Q}) \times H_2(M, \mathbb{Q}) \to \mathbb{Q}$, regardless of its singularity. We denote this signature $\sigma(M)$.

8.3.1. Eigenspace signatures. Working with complex coefficients, there is a similarly defined Hermitian pairing, $H_2(M, \mathbb{C}) \times H_2(M, \mathbb{C}) \to \mathbb{C}$. If $T\colon M \to M$ is a diffeomorphism of period p, then $H_2(M, \mathbb{C})$ splits into ζ^i eigenspaces, where ζ is a primitive p–root of unity. Call these eigenspaces H_{ζ^i}.

It is immediate that the eigenspaces are orthogonal with respect to the intersection form. We define the signature of the interesection form restricted to H_{ζ^i} by $\sigma_{\zeta^i}(M)$. The notation suppresses T; it will only be used when the transformation T is known.

8.4. Bordism

8.4.1. Bordism Groups. Closed n–manifolds M_1 and M_2 are called bordant if there is a compact $(n{+}1)$–manifold W with $\partial W \cong M_1 \cup -M_2$ (disjoint union). Bordism induces an equivalence relation on the set of closed manifolds, and the set of equivalence classes is denoted Ω_n.

Disjoint union induces an operation on the set of bordism classes. Under this operation Ω_n is an abelian group, with identity represented by either S^n, or, if the empty set is a manifold, then ϕ.

Exercise 8.4.1. Verify that the operation is also induced by connected sum.

We will need the following identifications of the low-dimensional bordism groups. For $n \le 2$ the result is immediate (knowing the classification of surfaces). For $n = 3$ and $n = 4$ the result is deeper and the reader is referred to the references.

Theorem 8.4.2. $\Omega_0 \cong \mathbb{Z}$; $\Omega_n \cong 0$ for $n = 1, 2, 3$; and $\Omega_4 \cong \mathbb{Z}$. The last isomorphism is given by the signature function.

8.4.2. Bordism of a Space. For a space X the bordism group $\Omega_n(X)$ consists of bordism equivalence classes of pairs (M, f) where M is a closed n–manifold and f is a map of M to X. Two such pairs (M_1, f_1) and (M_2, f_2) are bordant if there is a pair (W, F)

with W a bordism from M_1 and M_2 and $F\colon W \to X$, restricts to its boundary to give f_1 and f_2.

Disjoint union makes $\Omega_n(X)$ an abelian group. It is isomorphic to $\oplus\Omega_n(X_i)$, where $\{X_i\}$ si the set of path components of X.

Exercise 8.4.3. For a path connected space X, define a homomorphism $\Omega_n \to \Omega_n(X)$. Show that your map is injective. Show that for X a point (denoted $*$ henceforth), or more generally, if X is a contractible, then $\Omega_n \to \Omega_n(X)$ is an isomorphism.

Exercise 8.4.4. Define the relative bordism groups $\Omega_n(X, Y)$ where $Y \subset X$. Show that bordism is a generalized homology theory; that is, show that it is functorial with respect to continuous maps between pairs of spaces and that is satisfies the homotopy, excision, and long exact sequence axioms.

The following theorem uses the language of spectral sequences. All that we need from the theorem is the result stated in the subsequent corollary.

Theorem 8.4.5. For a CW-complex X, there is a homology spectral sequence converging to $\Omega_*(X)$ with E_2 term given by $E_2^{i,j} = H_i(X, \Omega_j)$.

Theorem 8.4.6. For a CW-complex X, $\Omega_i(X) \cong H_i(X)$ for $i \le 3$. For $n = 4$, if X is connected we have a split short exact sequence $\Omega_4 \to \Omega_4(X) \to H_4(X)$. The splitting is given by mapping X to a point. In particular, $\Omega_4 \cong H_4(X) \oplus \mathbb{Z}$ with the map to $\mathbb{Z}$ given by signature.

8.4.3. Bordism of a Group. For a group G we have the Eilenberg-MacLane space $K(G, 1)$. We define $\Omega_n(G) = \Omega_n(K(G, 1))$. In addition to the trivial group, the principal cases of interest to us will be ones laid out in the following theorem.

Theorem 8.4.7.

(1) $\Omega_n(\mathbb{Z}) \cong \mathbb{Z}$ if $n = 0, 1, 4$ and $\Omega_n(\mathbb{Z}) = 0$ if $n = 2, 3$.

(2) $\Omega_0(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}$; $\Omega_1(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$; $\Omega_2(\mathbb{Z}/p\mathbb{Z}) \cong= 0$; $\Omega_3(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$; $\Omega_4(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}$.

The properties of the 3-dimensional bordism groups follow most easily from a consequence of the bordism spectral sequence: $\Omega_3(G) = H_3(G)$. Of course they follow as well from more general bordism theory; [**?**] is a good reference.

To see that $\Omega_3(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, note that for a pair (M, χ) with $\chi\colon H_1(M) \to \mathbb{Z}/p\mathbb{Z}$ we view $\chi \in H^1(M, \mathbb{Z}/p\mathbb{Z})$. The quantity $\chi \cdot b(\chi)([M]_p)$ is the desired element in $\mathbb{Z}/p\mathbb{Z}$. Here b represents the Bockstein, $b\colon H^1(M, \mathbb{Z}/p\mathbb{Z}) \to H^2(M, \mathbb{Z}/p\mathbb{Z})$, the product is the cup product, and $[M]_p$ is the $\mathbb{Z}/p\mathbb{Z}$ reduction of the fundamental class of M.

A useful alternative definition of the isomorphism is given using the linking form, $\beta\colon \text{torsion}(H_1(M)) \times \text{torsion}(H_1(M)) \to \mathbb{Q}/\mathbb{Z}$. The restriction of the homomorphism χ to $\text{torsion}(H_1(M))$ is given by linking with some $x \in \text{torsion}(H_1(M))$; that is, $\chi(y) = \beta(x, y)$ for all $y \in \text{torson}(H_1(M))$. The self-linking of x, $\beta(x, x)$, is in $\mathbb{Q}/\mathbb{Z}$, but since it is p-torsion, it can be viewed as an element in $\mathbb{Z}/p\mathbb{Z}$.

We have the following result:

Theorem 8.4.8. If $|H_1(M)| = pm$ with p and m relatively prime, then for any non-trivial $\mathbb{Z}/p\mathbb{Z}$ character, $[M, \chi]$ is nonzero in $\Omega_3(\mathbb{Z}/p\mathbb{Z})$.

Proof For such a manifold the Bockstein is an isomorphism. By Poincaré duality the cup product is nontrivial in $H^3(M, \mathbb{Z}/p\mathbb{Z})$. Hence, $\chi \cdot b(\chi)$ is nontirvial in $H^3(M, \mathbb{Z}/p\mathbb{Z})$ and the result follows. $\square$

We also will be using the fact that map $\Omega_3(\mathbb{Z}/p\mathbb{Z}) \to \Omega_3(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z})$ induced by inclusion is an isomorphism, with inverse induced by projection. This follows from either the Kunneth formula on homology or Kunneth results on bordism. Again, see [**?**].

8.5. Casson-Gordon Invariants as Bordism Invariants

If the classes of $(M_1, \bar{\chi}_1)$ and $(M_2, \bar{\chi}_2)$ represent the same element in the bordism group $\Omega_3(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z})$, then $\tau(M_1^3, \bar{\chi}_1)$ and $\tau(M_2^3, \bar{\chi}_2)$ differ by an element in $W(\mathbb{C}(t))$. Hence the difference $\sigma(M_1, \bar{\chi}_1) - \sigma(M_2, \bar{\chi}_2)$ is an integer. It follows that σ defines a homomorphism $\sigma'\colon \Omega_3(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$. This homomorphism takes values in $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, and so can be viewed as a homomorphism $\sigma_p\colon \Omega_3(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}) \to \mathbb{Z}/p\mathbb{Z}$. The induced homomorphism on $\Omega_3(\mathbb{Z}/p\mathbb{Z})$ will also be denoted σ_p.

Theorem 8.5.1. For p odd, σ_p is an isomorphism.

Proof. A calculation of [**?**] shows that for the lens space $L(p, 1)$ given by p-surgery on the unknot in S^3 with $\mathbb{Z}/p\mathbb{Z}$ character taking value 1 on the meridian, $\sigma_p = 2$. Since σ_p is a homomorphism from $\mathbb{Z}/p\mathbb{Z}$ to itself, and p is odd, the result follows. $\square$

THEOREM 8.5.2. For a knot K and $\mathbb{Z}/p\mathbb{Z}$ character χ on $H_1(M_K)$, $\sigma_p(M_{K,0},\bar{\chi}) = \sigma_p(M_K,\chi)$. Equivalently, $p\sigma(K,\chi) = \sigma_p(M_K,\chi)$ mod p.

Proof. Since $\Omega_3(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}) \to \Omega_3(\mathbb{Z}/p\mathbb{Z})$ induces an isomorphism, it follows that $\sigma_p(M_{K,0},\bar{\chi}) = \sigma_p(M_{K,0},\chi \oplus (0))$, where (0) represents the trivial character to $\mathbb{Z}$. A $\mathbb{Z}/p\mathbb{Z}$-bordism from $(M_{K,0},\chi \oplus (0))$ to (M_K,χ) is constructed from $M_K \times [0,1]$ by adding a 2-handle with 0 framing to $M_K \times 1$. Note that χ extends over this bordism since the 2-handle is added along a null homologous curve (the lift of K), and provides the desired $\mathbb{Z}/p\mathbb{Z}$-bordism. □

We can now make one of our key observations.

THEOREM 8.5.3. If $H_1(M_K) = pm$ with $\gcd(p,m) = 0$ and p odd, then for any nontrivial $\chi\colon H_1(M_K) \to \mathbb{Z}/p\mathbb{Z}$, $\sigma(K,\chi) \neq 0$.

Proof. Note that $(M_K,\chi) \neq 0 \in \Omega_3(\mathbb{Z}/p\mathbb{Z})$. The linking form $\beta\colon H_1(M_K) \times H_1(M_K) \to \mathbb{Q}/\mathbb{Z}$ is a *nonsingular* pairing. Since the p-torsion is generated by a single element x, $\beta(x,x) \neq 0$, implying that for the character given by linking with x the $\mathbb{Z}/p\mathbb{Z}$-bordism class is nontrivial. □

Similarly we can prove:

THEOREM 8.5.4. If $\chi\colon H_1(M_K) \to \mathbb{Z}/p^r\mathbb{Z}$ is a character obtained by linking with the element $x \in H_1(M_K)$, then $\sigma(K,\chi) \equiv \beta(x,x)$ modulo $\mathbb{Z}$.

and a simple corollary, using the nonsingularity of β:

COROLLARY 8.5.5. If $H_1(M_K) = \mathbb{Z}/p^n\mathbb{Z} \oplus G$, and $k \in \mathbb{N}$ is such that $k > n/2$, $p^k \not| |G|$, and χ maps onto $\mathbb{Z}/p^k\mathbb{Z}$, then $\sigma(K,\chi) \neq 0$.

8.6. Knot signatures from 4–manifolds

Denote 0–surgery on a knot $K \subset S^3$ by $M(K,0)$. Recall, this manifold is formed from the knot exterior $X(K)$ by attaching a solid torus, interchanging the longitude and meridian. It satisfies $H_1(M(K,0)) \cong \mathbb{Z}$.

There is the canonical homomorphism $\epsilon\colon \pi_1(M(K,0)) \to \mathbb{Z}$ and since $\Omega_3(\mathbb{Z}) = 0$, there is a compact 4–manifold W with a homomorphism $\eta\colon \pi_1(W) \to \mathbb{Z}$ with $\partial W = M(K,0)$ and η extending ϵ. We can assume that $H_1(W) \cong \mathbb{Z}$ by surgering any other generators of H_1.

For a fixed integer p, let $\tilde{W}_p$ denote the p–fold cyclic cover of W, let $\sigma_{i/p}(W)$ denote the signature of the intersection form of the ω_p^i–eigenspace of $H_2(\tilde{W}_p, \mathbb{C})$, and let $\sigma(W)$ denote the signature of the intersection form of W.

DEFINITION 8.6.1. $\sigma_{i/p}(K) = \sigma_{i/p}(W) - \sigma(W)$.

THEOREM 8.6.2. $\sigma_{i/p}(K)$ is well-defined, independent of the choice of W.

Proof Suppose that (W', η') is a second bounding pair for $(M(K,0), \epsilon)$. Form the boundary union $Z = W \cup -W'$. By additivity of signatures (and eigenspace signatures) under boundary union the result will follow from the observation that $\sigma_{i/p}(Z) - \sigma(Z) = 0$.

By forming the connected sum of Z with copies of $\mathbb{C}P^2$ we can arrange that Z has signature 0. A direct calculation of intersection forms for $\mathbb{C}P^2$ shows that $\sigma_{i/p}(\mathbb{C}P^2) - \sigma(\mathbb{C}P^2) = 0$, and thus we haven't changed the value of the difference, $\sigma_{i/p}(Z) - \sigma(Z) = 0$. (The representation η restricted to $\mathbb{C}P^2$ is trivial, and thus the cover is trivial p–fold cyclic cover, consisting of p copies of $\mathbb{C}P^2$.) Since the signature of Z is trivial, Z represents $0 \in \Omega_4(\mathbb{Z})$. Thus, it bounds a 5–manifold Y^5 over $\mathbb{Z}$. As we saw earlier, this implies that $\sigma(Z) = 0$. A similar argument to that one also shows that $\sigma_{i/p}(Z) = 0$. □

THEOREM 8.6.3. If K is concordant to K', and p is a prime power, then $\sigma_{i/p}(K) = \sigma_{i/p}(K')$.

Proof Let A be a concordance between K and K'. That is, A is an annulus embedded in $S^3 \times I$. A neighborhood of A, diffeomorphic to $S^1 \times I \times I$ can be removed from $S^3 \times I$ and be replaced with $S^1 \times B^2 \times I$ to build a $\mathbb{Z}$–bordism between $M(K,0)$ and $M(K',0)$. Call this bordism W'' and note that $H_2(W'') = 0$. Furthermore, $H_2(\tilde{W}''_p, \mathbb{C}) = 0$ as long as p is a prime power. Thus, the manifold W' used to compute $\sigma_{i/p}(K')$ can be built from the manifold W used to compute $\sigma_{i/p}(K)$ by adding W'' along the boundary. This doesn't change any of the signatures involved. □

THEOREM 8.6.4. For a knot K, we have $\sigma_{i/p}(K) = \sigma_\omega(K)$, where $\omega = e^{2\pi i/p}$, and σ_ω is as defined in 2.10.13.

Proof See [**?**, **?**]. □

CHAPTER 9

Order Four in the Algebraic Concordance Group

Since the definition of the classical knot concordance group defined in 1966 by Fox and Milnor, one of the most vexing questions concerning the concordance group has been whether it contains elements of finite order other than 2-torsion. Interest in this question was heightened by Levine's proof that the algebraic concordance group contains an infinite summand generated by elements of order 4. In this chapter we outline the arguments of [**?**, **?**, **?**] to show that many knots from these algebraic concordance order 4 classes do not have concordance order 4. In proofs we use Casson-Gordon invariants coming from the two-fold cover M_K of S^3 branched over a knot K, and the bordism approach from Section 8.5. Nonlinearity of the invariants plays a crucial role.

To simplify notation, for any abelian group H and a prime p, let A_p denote the p-primary subgroup of H that contains elements with orders a power of the prime p.

9.1. $\mathbb{Z}/p\mathbb{Z}$ in homology

Recall from Section 6.3 that if $|H_1(M_K)_p| = p$, for a prime p, such that $p \equiv 3 \bmod 4$ and $\gcd(m, 3) = 1$, then K is of order 4 in $\mathcal{G}$. We will show here that any such K has infinite order in $\mathcal{C}$. We will first consider two simple examples.

9.1.1. Case $p = 3$: Suppose that $|H_1(M_K)| = 3m$, where $\gcd(m, 3) = 1$. We have that $H_1(M_K)_3$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$, generated by an element x with $\beta(x, x) = \pm(\frac{1}{3}) \in \mathbb{Q}/\mathbb{Z}$. Suppose now that K is of order d in $\mathcal{C}_3$. Any nontrivial metabolizing element for $H_1(M_{dK})_3 \cong (\mathbb{Z}/3\mathbb{Z})^d$ is of the form $(x_i)_{i=1\ldots d}$, where $x_i = \pm x$ for r values of i and is 0 otherwise. The self-linking of this element is 0, and by the Casson–Gordon Theorem 8.2.1 we have $\sigma(\# dM_K, (\chi_{x_i})_{i=1\ldots d}) = 0$ with exactly r of the $x_i = \pm 1$ and the rest 0. Here χ_y denotes the character given by linking with y.

Now, applying Gilmer's additivity theorem 8.2.2 we have that $r\sigma(K, \chi_1) = 0$. (We have used that $\sigma(M_K, 0) = 0$ and $\sigma(M_K, \chi_{-1}) = \sigma(M_K, \chi_1)$.) It follows of course that $\sigma(K, \chi_1) = 0$, contradicting 8.5.3.

9.1.2. Cae $p = 7$: Next, let $|H_1(M_K)| = 7m$,where 7 does not divide m. For such a knot K, a metabolizer for $H_1(M_{4K})_7 \cong (\mathbb{Z}_7)^4$ can be seen to be generated by a pair of elements $< (1, 0, 2, 3), (0, 1, -3, 2) >$ (There are other possibilities differing only in order and sign from this one.) Denoting by χ_a the $(\mathbb{Z}/7\mathbb{Z})$-character that takes value a on fixed generator of $H_1(M_K)_7$, we find from either of these metabozing vectors that $\sigma(M_K, \chi_1) + \sigma(M_K, \chi_2) + \sigma(M_K, \chi_3) = 0$, something that may be true. However, adding the generators we see that the metabolizer must also contain the vector $(1, 1, 6, 5)$ and its multiples, $(2, 2, 5, 3)$ and $(3, 3, 4, 1)$. From this we get the relations $3\sigma(M_K, \chi_1) + \sigma(M_K, \chi_2) = 0$, $3\sigma(M_K, \chi_2) + \sigma(M_K, \chi_3) = 0$, and $3\sigma(M_K, \chi_3) + \sigma(M_K, \chi_1) = 0$. Combining these one finds that $28\sigma(M_K, \chi_1) = 0$ again contradicting Theorem 8.5.3.

An interesting observation about the above argument is that the result does not follow from knowing the the vanishing of the Casson-Gordon invariants for a spanning set of metabolizing elements, or even their multiples. This demonstrates the very nonlinear property of these invariants and is the first application we know of in which that nonlinearity plays such an essential role. As in the case of $p = 3$ this obstructs knots of algebraic order 4 from being of order 4 in $\mathcal{C}$. For instance,

EXERCISE 9.1.1. Recall that the order of the 2-fold branched cover of a knot is given by $|\Delta(-1)|$, where Δ is the Alexander polynomial. If $\Delta(t) = nt^2 - (2n + 1)t + n$ with $n = 5 \bmod 7$ but $n \neq 12 \bmod 49$, then K is not of order 4 in $\mathcal{C}$.

By 6.3.1, any higher dimensional knot with such a polynomial is of order 4 in concordance.

9.1.3. The general case.

THEOREM 9.1.2. Let K be a knot in S^3 with 2-fold branched cover M_K. If the order of the first homology with integer coefficients satisfies $|H_1(M_K)| = pm$ with p a prime congruent to 3 mod 4 and $\gcd(p, m) = 1$, then K is of infinite order in the classical knot concordance group, $\mathcal{C}$.

Proof. Suppose that dK is slice. The existence of a $\mathbb{Z}/p\mathbb{Z}$-metabolizer implies that d is a multiple of 4. (The linking form of $H_1(M_K)$ represents an element of order 4 in the Witt group of $\mathbb{Z}/p\mathbb{Z}$ linking forms.) We begin by setting up some formalism to simplify the sort of linear algebra that appeared in the previous section. The example below illustrates the notation we develop next.

Any metabolizing vector for the linking form on $H_1(M_K, \mathbb{Z}/p\mathbb{Z})$ (in the subgroup L_p given by Theorem 5.1) can be written as $x = (x_i)_{i=1\ldots d} \in (\mathbb{Z}/p\mathbb{Z})^d$. The condition that a corresponding Casson-Gordon invariant vanishes yields $\sum_{x_i \neq 0} \sigma(M_K, \chi_{x_i}) = 0$. Now the x_i are in the cyclic group of nonzero elements in $\mathbb{Z}/p\mathbb{Z}$. Denoting a generator for this group by g, each nonzero x_i corresponds to g^{α_i} for some α_i. If we introduce further shorthand, setting $t^{\alpha_i} = \sigma(M_K, \chi_{x_i})$, we find that each metabolizing vector leads to a relation $\sum_{x_i \neq 0} t^{\alpha_i} = 0$. Note that at this point the symbol t^α does not represent a power of any element "t", it is purely symbolic. However it does permit us to view the relations as being elements in the ring $\mathbb{Z}[\mathbb{Z}/(p-1)\mathbb{Z}]$. Furthermore, since $\sigma_{x_i} = \sigma_{p-x_i}$, we have that $t^j = t^{j+(p-1)/2}$. (Recall that $g^{(p-1)/2} = -1$.) Hence, we can view the relations as sitting in $\mathbb{Z}[\mathbb{Z}/q\mathbb{Z}]$, where $q = (p-1)/2$.

Suppose that the metabolizing vector x corresponds to the relation $f = 0$, where f is represented by an element in $\mathbb{Z}[\mathbb{Z}/q\mathbb{Z}]$. Then a calculation shows that ax corresponds to the relation $t^\alpha f$ where $g^\alpha = a$. Hence it follows that the relations between Casson-Gordon invariants generated by a given element $x \in L_p$ and its multiples forms an ideal in $\mathbb{Z}[\mathbb{Z}/q\mathbb{Z}]$ generated by the polynomial f. Before applying this to complete the proof, we should pause for an example.

Example 9.1.3. Consider the metabolizing vector $x = (2, 3, 15, 16)$ in $(\mathbb{Z}/19\mathbb{Z})^4$. The nonzero elements of $\mathbb{Z}/19\mathbb{Z}$ are generated by 2, and we have $2 = 2^1$, $3 = 2^{13}$, $15 = 2^{11}$, and $16 = 2^4$. Hence in the notation just given, the vanishing of the corresponding Casson-Gordon invariant can be written as $t^1 + t^{13} + t^{11} + t^4 = 0$. Here we are in $\mathbb{Z}[\mathbb{Z}/18\mathbb{Z}]$. Switching to $\mathbb{Z}[\mathbb{Z}/9\mathbb{Z}]$ we have that $t^1 + t^4 + t^2 + t^4 = 0$. (Notice that χ_{15} and χ_4 yield the same Casson-Gordon invariant, and that χ_{15} corresponds to t^{11} while χ_4 corresponds to t^2 (since $2^2 = 4$) and $t^{11} = t^2 \in \mathbb{Z}[\mathbb{Z}/9\mathbb{Z}]$.)

Now consider the metabolizing vector $5x = (10, 15, 18, 4)$. Since $10 = 2^{17}, 15 = 2^{11}, 18 = 2^9$, and $4 = 2^2$, all mod 19, the corresponding relation in $\mathbb{Z}[\mathbb{Z}/18\mathbb{Z}]$ is $t^{17} + t^{11} + t^9 + t^2 = 0$. Reducing to $\mathbb{Z}[\mathbb{Z}/9\mathbb{Z}]$ gives $t^8 + t^2 + 1 + t^2$.

Hence by multiplying x by 5 we have gone from the equation $t^1 + t^4 + t^2 + t^4 = 0$ to $t^8 + t^2 + 1 + t^2 = 0$. Notice that the second polynomial is obtained from the first by multiplication by t^7. Finally $5 = 2^{16}$ mod 19, and $t^{16} = t^7 \in \mathbb{Z}[\mathbb{Z}/9\mathbb{Z}]$.

To return to the proof, we must analyze the possible metabolizers L_p for $(\mathbb{Z}/p\mathbb{Z})^{4k}$, where $d = 4k$. Such a metabolizer must be generated by $2k$ elements. Applying the

Gauss-Jordan algorithm to a basis for L_p, and perhaps reordering, we find a generating set $\{v_i\}_{i=1\ldots 2k}$ where the first $2k$ components of v_i are 0, except the i-component which is 1. Summing this basis produces the element $(1, 1, \ldots, 1, a_1, \ldots, a_{2k}) \in L_p$ where the first $2k$ entries are 1 and the a_i are unknown.

The corresponding relation is of the form $f = 2k + \sum_{i=1}^{k'} t^{\alpha_i} = 0$. (The sum may not contain $2k$ terms if any of the $a_i = 0$; hence k' is less than or equal to $2k$.) We next show that the ideal generated by f in $\mathbb{Z}[\mathbb{Z}/q\mathbb{Z}]$ contains a nonzero integer. This will follow from the fact that f and $t^q - 1$ are relatively prime, which will be the case unless f vanishes at some q-root of unity, ω; however, by considering norms and the triangle inequality we see that this will be the case only if $k' = 2k$ and $\omega^{a_i} = -1$ for all i. But since q is odd, no power of ω can equal –1.

Since we now have that f and $t^q - 1$ are relatively prime, it follows that with $\mathbb{Q}$ coefficients (so that we are working over a PID) there is a polynomial g satisfying $gf = 1$ mod $(t^q - 1)$. Clearing denominators we find that for some integral polynomial h, $hf = n$ mod $(t^q - 1)$ for some positive integer n.

The proof of the theorem is concluded by observing that we now have the relation corresponding to $n \in \mathbb{Z}[\mathbb{Z}/q\mathbb{Z}]$. That is, $n\sigma(M_K, \chi_1) = 0$. As before, this would imply that $\sigma(M_K, \chi_1) = 0$, contradicting Theorem 8.5.3. □

9.1.4. Low crossing number knots. Based on the work of Levine, Morita [**?**] developed an algorithm to determine the order of a knot in the algebraic concordance group using only its Alexander polynomial. Based on this, he enumerated all prime knots of 10 or fewer crossings that are of algebraic order 4. There are eleven such knots, including 7_7, 9_{34}, and nine 10 crossing knots. Of these, seven have $H_1(M)$ satisfying our criteria for $p = 3$. Three more satisfy the condition for $p = 7$, and the last, 10_{86}, has $H_1(M) = \mathbb{Z}/83\mathbb{Z}$.

9.1.5. Polynomial conditions. Recall that by Theorem 6.3.1 a knot K with $\Delta_K(t)$ quadratic is of finite order if $\Delta_K(t) = at^2 - (1 + 2a)t + a$ for some $a > 0$, and in that case it is of order 4 if for some prime $p = 3$ mod 4, $\Delta_K(-1) = p^\alpha m$ with α odd and $\gcd(p, m) = 1$. The above theorem applies only in the case that α can be assumed to be 1, however, that is sufficient to give an infinite family of examples, beginning with $\Delta_K(t) = 5t^2 - 11t + 5$, where $\Delta_K(-1) = (3)(7)$.

9.1.6. Infinitely many linearly independent examples. Knots formed as twisted doubles of the unknot were among the first knots used to construct algebraically slice knots that are not slice [**?, ?**]. Jiang [[**?**] used these knots to demonstrate that the set of algebraically slice knots contains a infinite set of knots that is linearly independent in concordance. Here we demonstrate that twisted doubles also provide such independent families of knots that are of algebraic order 4.

To achieve independence within a family of examples, our theorem must be extended somewhat. Here is the statement we need.

THEOREM 9.1.4. If $|H_1(M_K)| = pm$ with p a prime congruent to 3 mod 4 and $\gcd(p, m) = 1$, and if J is any knot with $|H_1(M_J)| = q$, where $\gcd(q, p) = 1$, then $dK\#J$ is not slice for all nonzero integers d.

Proof. If we consider $\mathbb{Z}/p\mathbb{Z}$ characters on the 2-fold branched cover, the characters all vanish on $H_1(M_J)$ so by the additivity of Casson-Gordon invariants we are reduced to considering the character restricted to $H_1(M_{dK})$, which places us in the setting of the proof of Theorem 9.1.2.

To apply this, let K_n denote the ($-n$)-twisted double of the unknot, with $n > 0$. Then $\Delta_{K_n}(t) = nt^2 - (1 + 2n)t + n$, and $H_1(M_{K_n}) = \mathbb{Z}/(4n + 1)\mathbb{Z}$. To pick an appropriate set of these knots, let $\{p_i\}$ be an enumeration of the primes that are congruent to 3 mod 4. Let $n_i = (p_{2i-1}p_{2i} - 1)/4$. Then the previous theorem quickly yields the following.

COROLLARY 9.1.5. The subset of the set of twisted doubles of the unknot given by $\{K_{n_i}\}$, is a linearly independent set in the concordance group and consists only of knots of algebraic order 4.

9.2. Prime-power order cyclic groups in homology

Next, we want to show that Theorem 9.1.2 can, in fact, be generalized to the case when $H_1(M_K)$ is cyclic with order a power of a prime that is congruent to 3 modulo 4. Once again, this rules out several algebraic order 4 knots from being concordance order 4.

THEOREM 9.2.1. Let K be a knot in S^3 with 2-fold branched cover M_K. If $H_1(M_K) = \mathbb{Z}/p^n\mathbb{Z} \oplus G$ with p a prime congruent to 3 mod 4, n odd, and p not dividing the order of G, then K is of infinite order in $\mathcal{C}_1$.

Proof Let K be a knot in S^3 with the 2-fold branched cover M_K. Suppose that $H_1(M_K) = \mathbb{Z}_{p^n} \oplus G$ with p a prime congruent to 3 mod 4, n odd, and p not dividing the order of G. We want to show that K is of infinite order in $\mathcal{C}$. The linking form of $H_1(M_K)$ represents an element of order 4 in the Witt group of $\mathbb{Z}_p$ linking forms. (See Corollary 23 (c) in [?].) If dK is slice, for some $d > 0$, then K is of concordance order d. Since Levine's homomorphism maps it to an order 4 element, it follows that $d = 4k$, for some positive integer k. We must analyze the possible metabolizers L_p for $(\mathbb{Z}_{p^n})^{4k}$.

A vector in L_p can be written as $x = (x_i)_{i=1\ldots d} \in (\mathbb{Z}_{p^n})^d$. Applying the Gauss-Jordan algorithm to a basis for L_p, and perhaps reordering, we can find a generating set of a particularly simple form. The next example illustrates a possible form for one such generating set, where the generators appear as the rows of the matrix.

Example 9.2.2. Let $H_p = (\mathbb{Z}_{p^3})^8$. A generating set for some metabolizer L_p of the standard nonsingular $\mathbb{Q}/\mathbb{Z}$ linking form can be written as follows:

$$\begin{pmatrix} 1 & * & * & * & * & * & * & * \\ 0 & p & 0 & 0 & * & * & * & * \\ 0 & 0 & p & 0 & * & * & * & * \\ 0 & 0 & 0 & p & * & * & * & * \\ 0 & 0 & 0 & 0 & p^2 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & p^2 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & p^2 & * \end{pmatrix}$$

In the above matrix, there is 1 row corresponding to p^0, and 3 rows each for p^1, and p^2.

We will denote the number of rows corresponding to p^i by k_i, the vectors in these k_i rows by $v_{i,1}, \cdots, v_{i,k_i}$, and $\sum_{j=0}^{i} k_j$ by S_i. Then, in general, the generating set consists of $\{v_{i,j}\}_{i=0,\ldots,n-1,j=1,\ldots,k_i}$ where $0 \le k_i \le 2k$, such that the first S_i entries of $v_{i,j}$ are 0, except for the $S_{i-1} + j$ entry which is p^i, and each of the remaining entries is divisible by p^i. From 2.7 it follows that $k_i = k_{n-i}$, for $i > 0$, and $S_{(n-1)/2} = 2k$.

If $a \in H_1(M_k)_p$, let $\chi_a \colon H_1(M_K) \to \mathbb{Q}/\mathbb{Z}$ be the character given by linking with a. In the case that $H_1(M_k)_p$ is cyclic, isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$, we can fix a generator of $H_1(M_k)_p$ and write χ_a where a is an integer representing an element in $\mathbb{Z}/p^n\mathbb{Z}$.

With this notation, we now see that our goal is to show that $\sigma(K, \chi_{p^{(n-1)/2}}) = 0$. Since $\chi_{p^{(n-1)/2}}$ maps onto $\mathbb{Z}/p^{(n+1)/2}\mathbb{Z}$ this will contradict 2.6 and it will follow that K cannot be of finite order.

As in Example 9.2.2, arrange the $\{v_{i,j}\}$ as rows of a $(4k - k_0) \times 4k$ matrix following the dictionary order on (i, j). We multiply the first k_0 vectors by p^{n-1}, the next k_1 vectors by p^{n-2}, and, so on, to obtain p^{n-1} on the diagonal. Clear the off-diagonal entries in the left $(4k - k_0) \times (4k - k_0)$ block. Now, adding all the rows gives us a vector in L_p with the first $4k - k_0$ entries equal to p^{n-1}. This vector corresponds to a character χ, given by linking an element with it, to $\mathbb{Z}/p\mathbb{Z}$ on which the Casson-Gordon invariants should vanish. That is, $\sigma((4k)K, \chi) = 0$. By 2.4 this leads to a relation of the form $(4k - k_0)\sigma(K, \chi_{p^{n-1}}) + \sum_{x_i \neq 0} \sigma(K, \chi_{x_i}) = 0$, where x_i are the remaining k_0 entries, each of which is divisible by p^{n-1}.

The set of nonzero characters from $\mathbb{Z}/p^n\mathbb{Z}$ to $\mathbb{Z}/p\mathbb{Z}$ is isomorphic to the multiplicative group of units in $\mathbb{Z}/p\mathbb{Z}$, which is a cyclic group of order $p - 1$. Denoting a generator for this group by g, each nonzero χ_{x_i} corresponds to g^{α_i} for some α_i. The correspondence can be given by $\chi_{x_i} \to g^{x_i/p^{n-1}}$. As in [LN] we use further shorthand, setting $t^{\alpha_i} = \sigma(K, \chi_{x_i})$. Each metabolizing vector leads to a relation $\sum_{x_i \neq 0} t^{\alpha_i} = 0$. Note that at this point the symbol t^α does not represent a power of any element "t", it is purely symbolic. However it does permit us to view the relations as being elements in the ring $\mathbb{Z}[\mathbb{Z}/(p-1)\mathbb{Z}]$. Furthermore, since $\sigma(K, \chi_{x_i}) = \sigma(K, \chi_{p^n - x_i})$, we have that $t^j = t^{j+(p-1)/2}$. (Recall that $g^{(p-1)/2} = -1$.) Hence, we can view the relations as sitting in $\mathbb{Z}[\mathbb{Z}/q\mathbb{Z}]$, where $q = (p-1)/2$.

If a metabolizing vector x corresponds to the relation $f = 0$, where f is represented by an element in $\mathbb{Z}[\mathbb{Z}/q\mathbb{Z}]$, then ax corresponds to the relation $t^\alpha f$ where $g^\alpha = a$. It follows that the relations between Casson-Gordon invariants generated by the element $x \in L_p$ together with its multiples form an ideal in $\mathbb{Z}[\mathbb{Z}/q\mathbb{Z}]$ generated by the polynomial f.

With this in mind our relation can be written as $f = (4k - k_0) + \sum_{i=1}^{k'} t^{\alpha_i} = 0$, where $k' \leq k_0$. (Note that $4k - k_0 = S_{n-1}$.) We show that the ideal generated by f in $\mathbb{Z}[\mathbb{Z}/q\mathbb{Z}]$ contains a nonzero integer. This will follow from the fact that f and $t^q - 1$ are relatively prime, which will be the case unless f vanishes at some q-th root of unity, say ω; however, by considering norms and the triangle inequality we see that this will be the case only if $k' = 2k$ and $\omega^{a_i} = -1$ for all i. But since q is odd, no power of ω can equal -1.

It follows that $n\sigma(K, \chi_{p^{n-1}}) = 0$, for some $n \in \mathbb{Z}$. This implies that $\sigma(K, \chi_{p^{n-1}}) = 0$. Similarly we can show that $\sigma(K, \chi_{ap^{n-1}}) = 0$, for $0 < a < p$.

Next, let l be a nonnegative integer, and assume that $\sigma(K, \chi_{ap^s}) = 0$, for all $a \in \mathbb{Z}$, and all s such that $l < s \leq n - 1$. We will show that $\sigma(K, \chi_{p^l}) = 0$.

For $0 \leq i \leq S_l$, we multiply the vectors from the $(S_{i-1} + 1)$st to the S_ith vector by p^{l-i}, clear off-diagonal entries in the upper left $S_l \times S_l$ square block, and add the first S_l rows to get a vector in L_p with first S_l entries equal to p^l, and the remaining entries divisible by p^l. Since we have assumed that $\sigma(K, \chi_{ap^s}) = 0$, for $l < s \leq n - 1$, we can ignore the entries which are of the form ap^s, with $s > l$. Then we have a character to the multiplicative group of units in $\mathbb{Z}/(p^{n-l})\mathbb{Z}$. Since p is odd, this is a cyclic group of order $p^{n-l-1}(p-1)$ (see [D]). Again, since $\sigma(K, \chi_{x_i}) = \sigma(K, \chi_{p^n - x_i})$, we can view the relations as sitting in $\mathbb{Z}[\mathbb{Z}/q\mathbb{Z}]$, where $q = p^{n-l-1}(p-1)/2$. As $p^{n-l-1}(p-1)/2$ is odd, as above, it follows that the relation $f = S_l + \sum_{i=1}^{k'} t^{\alpha_i} = 0$, where $0 \leq k' \leq 4k - S_l$, is relatively prime to $t^q - 1$. It follows that $\sigma(K, \chi_{p^l}) = 0$.

Thus, we have $\sigma(K, \chi_{p^{(n-1)/2}}) = 0$, which contradicts 8.5.5, and proves that K cannot be of finite order in the concordance group. □

COROLLARY 9.2.3. Let n be a positive integer such that some prime p congruent to 3 mod 4 has odd exponent in the prime power factorization of $4n+1$. Then a knot K with Alexander polynomial $nt^2 - (2n+1)t + n$ and H_p cyclic is of infinite order in the concordance group.

9.3. Examples

Although Theorem 9.2.1 applies to arbitrary knots and gives a simple test to see if a knot is of infinite order, it is of most interest in the case that the knot is of algebaic order 4. As we will summarize below, Levine's results point to the primes congruent to 3 mod 4 as being crucial in determining if a knot is of algebraic order 2 or 4. (In fact, a careful examination of Levine's result shows that in any case in which Theorem 1.2 applies to a knot of finite algebraic order, that order must be 4.)

Since Theorems 9.1.2 and 9.2.1 do apply to particular knots that are of algebraic order 4, we have as a consequence examples of algebraically slice knots that are not slice; simply consider the connect sums of four copies of that knot. The first examples of non-slice, algraically slice knots were the breakthrough achievement of [**?**, **?**]; the results there depended on the computation of subtle nonabelian invariants. But the obstruction described here depends only on one of the simplest abelian invariants. We begin by stating Levine's theorem (Corollary 23 in [**?**]) in the case of greatest interest to us.

THEOREM 9.3.1. If a knot K has quadratic Alexander polynomial $\Delta(t)$ then:

(a) K is of finite order in the algebraic concordance group if and only if $\Delta(1)\Delta(-1) < 0$, in which case K is of order 2 or 4.

(b) K is of order 4 in the algebraic concordance group if an only if for some prime $p > 0$ with $p \equiv 3 \bmod 4$, $\Delta(1)\Delta(-1) = -p^a q$ where a is odd and $q > 0$ is relatively prime to p.

It is a bit unexpected that Theorem 9.2.1 does not apply in all cases of order 4 knots given by Theorem 9.3.1. The issue is that the conditions of Theorem [?, Corollary 23] do not assure that the homology of the 2-fold cover is cyclic. The next example demonstrates this. It is the simplest such example in terms of the size of the entries and the coefficients of the Alexander polynomial. The complexity of this example illustrates the strength of Theorem 9.2.1. The example is obtained by letting K be a knot with Seifert form:

$$A = \begin{pmatrix} 45 & 10 \\ 11 & -49 \end{pmatrix}.$$

The Alexander polynomial for K is $\Delta(t) = -2315 - 4631t - 2315t^2$. We have that $\Delta(1) = 1$, $\Delta(-1) = -9261 = -3^3 * 7^3$, and hence by Theorem 23cLevine, K is of order 4 in algebraic concordance.

The homology of M_K is presented by $A + A^T$:

$$A = \begin{pmatrix} 90 & 21 \\ 21 & -98 \end{pmatrix}.$$

A simple manipulation shows that this presents the group $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/49\mathbb{Z}$. Because this is not cyclic, Theorem [?] does not apply. In the next section we extend our techniques to deal with the case of $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$ and show that any knot with this Seifert form is not of order 4 in concordance.

Infinite families of examples for which Theorem 9.2.1 does apply are easily obtained by beginning with knots with Seifert matrix of the form:

$$A = \begin{pmatrix} a & 1 \\ 0 & -b \end{pmatrix}.$$

where a and b are postive integers. The Alexander polynomial of such a knot is $\Delta(t) = -ab + (1 - 2a)t - abt^2$, and $\Delta(-1)\Delta(1) = -(4ab + 1)$. hence by Theorem 8.5.1 it is of finite order. By diagonalizing $V + V^t$, one finds that the homology of the 2-fold branched cover is $\mathbb{Z}_{(4ab+1)}$. Hence, for any a and b for which $4ab + 1$ has a prime factor that is equal to 3 mod 4 and which has odd exponent, the knot is of infinite order in concordance.

It is of some interest to give geometric conditions with assure infinite concordance order. Here is a simple one. If a knot can be unknotted by cutting out a ball that intersects the knot in two arcs and regluing via a homeomorphism of the boundary, then the homology of the 2-fold branched cover is cyclic. (The 2-fold cover is constructed from

S^3 by cutting along a torus and regluing.) Hence, Theorem 1.2 can be applied. Such knots include unknotting number one knots and 2-bridge knots. (See [?] or [?] for details concering 2-bridge knots.) A simple application is the following:

The 2-bridge knot $K(p,q)$ has infinite order in the knot concordance group if some prime congruent to 3 mod 4 has odd exponent in p.

As an example, for a and b positive, the 2-bridge knot $K(4ab+1, 2a)$ is of infinite order in the knot concordance group if some prime congruent to 3 mod 4 has odd exponent in $4ab+1$. (One can check that these give example of knots with Seifert matrices of the form

$$A = \begin{pmatrix} a & 1 \\ 0 & -b \end{pmatrix},$$

and hence are in fact included among those discussed above. However, because they are two bridge knots, it is unnecessary to find the Seifert matrix in order to determine that they are of infinite concordance order. The previous calculation does show that this class contains elements of finite order in algebraic concordance.

The methods of proof considered so far do not extend easily. We see this in the example of the next section which appeared in [?].

9.4. $H_1(M_K)_3 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3^{2i}\mathbb{Z}$

Now consider the case that $\Delta_K(t)$ is quadratic and $\Delta_K(-1) = 27m$ where 3 does not divide m. This is another case of an algebraic order 4 knot. If $H_1(M_K)$ is cyclic, then by Theorem 9.2.1, we can say that K cannot have order 4 in $\mathcal{C}$. However, a knot with this polynomial may have $H_1(M_K) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$. We deal with this next.

Suppose that K satisfies $H_1(M_K)_3 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3^{2i}\mathbb{Z}$ and the characters χ under consideration will take values in $\mathbb{Z}/3^{2i}\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$, and such χ factor through characters defined on $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3^{2i}\mathbb{Z}$. Any such character is given by linking with an element of the $H_1(M_K)_3$, say $(x,y) \in \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3^{2i}\mathbb{Z}$. To simplify notation we will write $\sigma(K,\chi)$ as $\sigma_{x,y}$.

Our main result is as follows.

Theorem 9.4.1. If $H_1(M_K)_3 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3^{2i}\mathbb{Z}$ then K is not of order 4 in $\mathcal{C}$.

We will also present applications of this result, describing new infinite families of knots that are of algebraic order four but do not represent 4–torsion in $\mathcal{C}$. A simple, easily stated application is the following, where the Alexander polynomial of a knot K is denoted $\Delta_K(t)$:

Corollary 9.4.2. If $\Delta_K(t)$ is quadratic and $\Delta_K(-1) = 27m$ where 3 does not divide m, then K is of order 4 in $\mathcal{G}$ but not in $\mathcal{C}$.

We will assume that $4K$ is slice and consider all possible metabolizers to the linking form on $(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3^{2i}\mathbb{Z})^4$ and show that each leads to a contradiction to Theorem 8.5.5.

Lemma 9.4.3. There is a generating set $\{v, w\}$ for $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3^{2i}\mathbb{Z}$ such that v is of order 3, w is of order 3^{2i}, and the linking form satisfies: $\beta(v, v) = \pm 1/3$, $\beta(w, w) = \pm 1/3^{2i}$, and $\beta(v, w) = 0$.

Proof Let a generate the $\mathbb{Z}/3\mathbb{Z}$ summand and let b generate the $\mathbb{Z}/3^{2i}\mathbb{Z}$ summand. Since there is a character to $\mathbb{Q}/\mathbb{Z}$ taking value $1/3^{2i}$ on b, by the nonsingularity of the linking form there is an element x satisfying $\beta(x, b) = 1/3^{2i}$. Write $x = ra + sb$. Since $\beta(a, b)$ is a multiple of $1/3$ (a is of order 3), $s\beta(b, b)$ must of the form $t/3^{2i}$ with t not divisible by 3. Hence there is an integer u such that $u\beta(b, b) = 1/3^{2i}$. Let $v = a - 3^{2i}\beta(a, b)ub$. It is easily checked that v is of order 3 and $\beta(v, b) = 0$.

By the nonsingularity of the linking form, $\beta(v, v) = \pm 1/3$. As observed above, $\beta(b, b) = t/3^{2i}$ for some $t \in \mathbb{Z}/3^{2i}\mathbb{Z}$, $t \not\equiv 0 \bmod 3$. Let s be the inverse to t in $\mathbb{Z}/3^{2i}\mathbb{Z}$. Then $\pm s = q^2$ for some $q \in \mathbb{Z}/3^{2i}\mathbb{Z}$. (The square of an element is 0 mod 3 if and only if the element itself is such. In $\mathbb{Z}/3^{2i}\mathbb{Z}$ there are a total of 3^{2i-1} elements which are 0 mod 3. It follows that there are $3^{2i} - 3^{2i-1}$ elements which are ± 1 mod 3, half of which are additive inverses of the other half, and there are $\frac{3^{2i}-3^{2i-1}}{2}$ distinct squares which are not 0 mod 3.) Let $w = qb$. □

From now on we will fix the generating set to be as given in the previous lemma.

In order to apply Theorem 8.2.1 to the knot $4K$, we let $H = H_1(M_{4K})_3 \cong (\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3^{2i}\mathbb{Z})^4 \cong (\mathbb{Z}/3\mathbb{Z})^4 \oplus (\mathbb{Z}/3^{2i}\mathbb{Z})^4$. We will let M denote a metabolizer in H. To set up notation, we will represent an element in $(\mathbb{Z}/3\mathbb{Z})^4 \oplus (\mathbb{Z}/3^{2i}\mathbb{Z})^4$ by an ordered 8–tuple and a collection of n elements in $(\mathbb{Z}/3\mathbb{Z})^4 \oplus (\mathbb{Z}/3^{2i}\mathbb{Z})^4$ by an $n \times 8$ matrix, the rows of which represent the individual elements. Each element will be written as

$$u_i = v_i \oplus w_i \in (\mathbf{Z}_3)^4 \oplus (\mathbf{Z}_{3^{2i}})^4, 1 \leq i \leq 4.$$

Lemma 9.4.4. Let M be a metabolizer for H. Then M cannot be generated by less than four elements.

Proof Tensor H and M with $\mathbb{Z}/3\mathbb{Z}$. We have $H \otimes \mathbb{Z}/3\mathbb{Z} \cong (\mathbb{Z}/3\mathbb{Z})^8$. If M is generated by k elements, then $M \otimes \mathbb{Z}/3\mathbb{Z} \cong (\mathbb{Z}/3\mathbb{Z})^k$. If $k \leq 3$, then $\mathrm{rk}((H \otimes \mathbb{Z}/3\mathbb{Z})/(M \otimes \mathbb{Z}/3\mathbb{Z})) \geq 5$.

As $\mathrm{rk}((H/M) \otimes \mathbb{Z}/3\mathbb{Z}) \geq \mathrm{rk}((H \otimes \mathbb{Z}/3\mathbb{Z})/(M \otimes \mathbb{Z}/3\mathbb{Z}))$, we have a contradiction to the fact that $H/M \cong M$. $\square$

We will call the minimum number of elements required to generate M, the rank of M. The proof of Theorem 1.2 is simplest in the case that the rank is greater than 4.

THEOREM 9.4.5. If $\mathrm{rank}(M) = k$, $k > 4$, then K is not of order 4 in concordance.

Proof Consider a minimal generating set $\{(v_i, w_i)\}_{i=1\ldots k}$. These form the rows of a $k \times 8$ matrix which we denote $(V|W)$, where V and W are each $k \times 4$. We will now perform row operations to simplify the generating set. It will be convenient to interchange columns in these matrices as well, but notice that if two columns of W are interchanged, the same columns of V will be interchanged, since these columns correspond to the homology of the cover of a given component of $4K$.

By performing row operations and column interchanges, W can be put in upper triangular form. Hence, the fifth row of W is the trivial vector, $(0,0,0,0) \in (\mathbb{Z}/3^{2i}\mathbb{Z})^4$. After further column swaps, the fifth row of V can be put in the form $(\pm 1, \pm 1, \pm 1, 0)$, as these are the only nontrivial elements in $(\mathbb{Z}/3\mathbb{Z})^4$ with trivial self–linking.

It follows that $3\sigma_{1,0} = 0$, and hence $\sigma_{1,0} = 0$. However, by Theorem 8.5.3, $\sigma_{1,0} \equiv 1/3 \bmod \mathbb{Z}$, giving a contradiction. $\square$

The rest of this section is devoted to the case that $\mathrm{rank}(M) = 4$.

LEMMA 9.4.6. Let $\mathrm{rank}(M) = 4$. Then M has a generating set $\{\, u_j = v_j \oplus w_j \in (\mathbb{Z}/3\mathbb{Z})^4 \oplus (\mathbb{Z}/3^{2i}\mathbb{Z})^4 \,|\, j = 1,2,3,4 \,\}$ such that the corresponding matrix $(V|W)$ is of the form given below. The $v_{i,j}$ are elements in $\mathbb{Z}/3\mathbb{Z}$ and the $w_{i,j}$ are elements in $\mathbb{Z}/3\mathbb{Z}^{2i}$.

$$\left(\begin{array}{cccc|cccc} v_{1,1} & v_{1,2} & v_{1,3} & v_{1,4} & 1 & 0 & w_{1,3} & w_{1,4} \\ v_{2,1} & v_{2,2} & v_{2,3} & v_{2,4} & 0 & 1 & w_{2,3} & w_{2,4} \\ v_{3,1} & v_{3,2} & v_{3,3} & v_{3,4} & 0 & 0 & 3^{2i-1} & 0 \\ v_{4,1} & v_{4,2} & v_{4,3} & v_{4,4} & 0 & 0 & 0 & 3^{2i-1} \end{array}\right)$$

Proof Row operations and column swaps (provided the same column swaps are made in V as in W) can be used to make W upper triangular with the diagonal entries non-decreasing powers of 3 such that the remaining entries in the jth row are annihilated by the same power of 3 as is the diagonal entry. Let the diagonal entries be 3^{k_j} with $0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq 2i$. It is easily seen that the order of the element u_j represented by row j of this matrix is 3^{2i-k_j} and together the u_j generate a subgroup of order

$3^{(8i-\sum k_j)}$. On the other hand, the order of H is 3^{8i+4} and M has the square root order 3^{4i+2}. It follows that $\sum k_j = 4i-2$.

We first note that $k_4 \neq 2i$: If $k_4 = 2i$ then the last row has the form

$$(v_{4,1}, v_{4,2}, v_{4,3}, v_{4,4} \mid 0, 0, 0, 0)$$

with some of the $v_{4,j}$ nonzero. Since the self–linking of this element is 0, exactly 3 of the entries would be nonzero and it would follow that $3\sigma_{1,0} = 0$, implying that $\sigma_{1,0} = 0$, contradicting Corollary 8.5.5.

Hence, we have $0 \le k_1 \le k_2 \le k_3 \le k_4 \le 2i-1$.

If $k_4 < 2i-1$, then the generator u_4 generates a cyclic subgroup of order greater than 3. As $\sum k_j = 4i-2$, k_4 cannot be zero. It follows that $H/\langle\, u_4 \,\rangle$ has rank 8. This implies that $H/\langle\, u_1, u_2, u_3, u_4 \,\rangle$ has rank 5 or more. However, as $H/M \cong M$ (see 8.2.1), the rank of H/M is 4. Therefore, we have $k_4 = 2i-1$, $0 \le k_1 \le k_2 \le k_3 \le 2i-1$, and $k_1 + k_2 + k_3 = 2i-1$. As k_3 cannot be 0 either, a similar argument shows that k_3 will have to be $2i-1$. Therefore we have $k_1 = k_2 = 0$. It is easy to see that the entries above the 1 in the second row and the 3^{2i} in the last row can be made 0. $\square$

Our argument continues to proceed by ruling out possible metabolizers under the assumption that $4K$ is slice.

Lemma 9.4.7. Each of the entries $w_{i,j}$ in $(V|W)$ in the form given by Lemma 9.4.6 may be assumed to be ± 1 mod 3. The $\mathbb{Z}/3\mathbb{Z}$ reductions of the elements $(0, 0, w_{1,3}, w_{1,4})$ and $(0, 0, w_{2,3}, w_{2,4})$ are linearly independent in $(\mathbb{Z}/3\mathbb{Z})^4$.

Proof The self–linking of the first row is computed to be $\frac{\alpha}{3} \pm \frac{(1+w_{1,3}^2+w_{1,4}^2)}{3^{2i}}$ where α is determined by the self-linking of the $v_{1,j}$. If either $w_{1,3}$ or $w_{1,4}$ were 0 mod 3 then it is easily shown that this sum could not be an integer; basically, 0 is not the sum of two nontrivial squares modulo 3. It follows that neither $w_{1,3}$ nor $w_{1,4}$ can be 0. A similar argument applies for $w_{2,3}$ and $w_{2,4}$.

If the elements $(0, 0, w_{1,3}, w_{1,4})$ and $(0, 0, w_{2,3}, w_{2,4})$ were dependent over $\mathbb{Z}/3\mathbb{Z}$, then by combining the first two rows of $(V|W)$ we would have $(*, *, *, * \mid \pm 1, \pm 1, 3a, 3b)$. But such an element cannot have self–linking 0. $\square$

Lemma 9.4.8. The metabolizer M contains an element of the type

$$(1, 1, *, * \mid 0, 0, 3^{2i-1}m, 3^{2i-1}n),$$

where m and n are integers.

Proof Let $v_{i,j},\ w_{i,j}$ and $u_i = v_i \oplus w_i$ be as in Lemma 9.4.6

Suppose that $(v_{3,1}, v_{3,2})$ and $(v_{4,1}, v_{4,2})$ are linearly dependent in $(\mathbb{Z}/3\mathbb{Z})^2$. Then a nontrivial combination of u_3 and u_4 would yield an element of the form

$$(0, 0, *, * \mid 0, 0, 3^{2i-1}m, 3^{2i-1}n) \in M.$$

Note that non-triviality in this case is over $\mathbb{Z}_3$. In other words, either m or n is nonzero mod 3. To have self–linking zero the $*$ entries would have to be 0, so that we have $u = (0, 0, 0, 0 \mid 0, 0, 3^{2i-1}m, 3^{2i-1}n) \in M$.

Now, from Lemma 9.4.7 $(w_{1,3}, w_{1,4})$ and $(w_{2,3}, w_{2,4})$ are linearly independent over $\mathbb{Z}_3$, so a linear combination of these yields a vector whose $\mathbb{Z}_3$ reduction is $(1, 0)$. As the corresponding linear combination of u_1, u_2 is an element in M and therefore links the above u trivially, we have $m \equiv 0 \bmod 3$. Similarly $n \equiv 0 \bmod 3$, giving us a contradiction.

It follows that $(v_{3,1}, v_{3,2})$ and $(v_{4,1}, v_{4,2})$ are independent over $\mathbb{Z}_3$. Now, by taking an appropriate combination of u_3 and u_4 we can find the desired element of M. □

LEMMA 9.4.9. For $a, b \in \{0, \pm 1\}$, M contains elements of the form

$$(1, 1, *, * \mid 3^{2i-1}a, 3^{2i-1}b, 3^{2i-1}m, 3^{2i-1}n),$$

where $m, n \in \mathbb{Z}$ and exactly one of the $*$ entries is nonzero.

Proof Add $3^{2i-1}a$ times the first row and $3^{2i-1}b$ times the second row of the matrix to the element given in the previous lemma. The condition on the first two $*$s comes from the fact that the self–linking of the resulting element must be 0. □

Proof of Theorem 9.4.1

By Theorem 8.5.4, $\sigma_{1,0}$, $\sigma_{1,3^{2i-1}}$ and $\sigma_{1,2\cdot 3^{2i-1}}$ are nonzero.

From the previous lemma we have, in the case $a = b = 0$, that either $3\sigma_{1,0} = 0$, $2\sigma_{1,0} + \sigma_{1,3^{2i-1}} = 0$ or $2\sigma_{1,0} + \sigma_{1,2\cdot 3^{2i-1}} = 0$. The possibility that $3\sigma_{1,0} = 0$ contradicts Corollary 8.5.5, so either $2\sigma_{1,0} + \sigma_{1,3^{2i-1}} = 0$, or $2\sigma_{1,0} + \sigma_{1,2\cdot 3^{2i-1}} = 0$.

Similarly, by letting $a = b = 1$ we have either $2\sigma_{1,3^{2i-1}} + \sigma_{1,0} = 0$ or $2\sigma_{1,3^{2i-1}} + \sigma_{1,2\cdot 3^{2i-1}} = 0$.

Finally, letting $a = b = -1$ we have either $2\sigma_{1,2\cdot 3^{2i-1}} + \sigma_{1,0} = 0$ or $2\sigma_{1,2\cdot 3^{2i-1}} + \sigma_{1,3^{2i-1}} = 0$.

Considering the two relations $2\sigma_{1,0} + \sigma_{1,3^{2i-1}} = 0$ and $2\sigma_{1,3^{2i-1}} + \sigma_{1,0} = 0$ together, it follows that $3\sigma_{1,0} = 0$, contradicting Corollary 8.5.5. Similar considerations with pairs of relations rule out several possibilities.

Only two possibilities remain: the first is that $2\sigma_{1,0}+\sigma_{1,3^{2i-1}}=0$, $2\sigma_{1,3^{2i-1}}+\sigma_{1,2\cdot 3^{2i-1}}=0$, and $2\sigma_{1,2\cdot 3^{2i-1}}+\sigma_{1,0}=0$; the second is that $2\sigma_{1,0}+\sigma_{1,2\cdot 3^{2i-1}}=0$, $2\sigma_{1,3^{2i-1}}+\sigma_{1,0}=0$, and $2\sigma_{1,2\cdot 3^{2i-1}}+\sigma_{1,3^{2i-1}}=0$. Either case quickly implies that $3^{2i}\sigma_{1,0}=0$, so $\sigma_{1,0}=0$, again contradicting 8.5.4 and 8.5.3. □

9.5. More examples

Consider a knot with Alexander polynomial $\Delta_K(t)=kt^2-(2k+1)t+k$, $k\geq 0$ According to [**?**, Corollary 23] such a knot has finite order in the algebraic concordance group. It will have algebraic concordance order 4 if and only if there is some prime congruent to 3 mod 4 which has odd exponent in $4k+1$. According to [**?**], if $4k+1=3m$ with m prime to 3 then K is not of order 4 in concordance. We have the following extension.

COROLLARY 9.5.1. If $\Delta_K(t)=kt^2-(2k+1)t+k$ and $4k+1=(3^{2n+1})m$ with $n=0$ or 1 and m prime to 3 then K is not of order 4 in concordance.

Proof The case $n=0$ is settled by [**?**]. So let $n=1$. Since the Alexander polynomial is quadratic, $H_1(M_K)$ is of rank at most 2. In the case that the rank is 1, then $H_1(M_K)_3\cong\mathbb{Z}_{27}$ and hence Theorem 9.2.1 applies to show that K is not of order 4. In the case that the rank of $H_1(M_K)_3$ is 2, then $H_1(M_K)_3\cong\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}_9$ and Theorem 9.4.1 applies.

Doubled Knots According to [**?, ?**] the k–twisted double of the unknot, D_k, is algebraically slice if and only if $4k+1=l^2$ for some integer l. We are thus interested in the case that $4k+1=9m^2$ with m prime to 3. For this to occur, m must be odd: $m=2n+1$. Solving gives $k=9(n^2+n)+2$. Furthermore, m will be prime to 3 if $n\neq 1$ mod 3.

A similar calculation shows that D_k satisfies $H_1(D_K)\cong\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}_m$ with m prime to 3 if $k=3n+2$ with $n\neq 0$ mod 3. Hence, we have the corollary:

COROLLARY 9.5.2. For all positive $r\neq 0$ mod 3 and positive $s\neq 1$ mod 3, the knot $D_{3r+2}\#D_{9(s^2+s)+2}$ is of algebraic order 4 but is not of order 4 in concordance.

Finally, these results apply to s-equivalence classes of knots. To show that the algebraic concordance class of a knot K cannot be realized by a knot of concordance order 4, we need to consider knots with the same Seifert form as $K\#J$, where J is algebraically slice. The present paper marks the first progress in that direction, by showing that if $H_1(M_K)\cong\mathbb{Z}/3\mathbb{Z}$ and that J is an algebraically slice knot with $H_1(M_J)_3\cong\mathbb{Z}/3^{2i}\mathbb{Z}$. Then $K\#J$ is not of order 4.

CHAPTER 10

Smooth and topological concordance

10.1. The topological category

In [?] Freedman developed surgery theory in the category of topological 4–manifolds, proving roughly that for manifolds with fundamental groups that are not too complicated (in particular, finitely generated abelian groups) the general theory of higher-dimensional surgery descends to dimension 4. The most notable consequence of this work was the proof the 4–dimensional Poincaré Conjecture: a closed topological 4–manifold that is homotopy equivalent to the 4–sphere is homeomorphic to the 4–sphere.

Two significant contributions to the study of concordance quickly followed from Freedman's original paper. The first of these, proved in [?], is that a locally flat surface in a topological 4–manifold has an embedded normal bundle. The use of such a normal bundle was implicit in the proof that slice knots are algebraically slice. It is also used in a key step in the proof of the Casson–Gordon theorem, as follows. Casson–Gordon invariants of slice knots are shown to vanish via the observation that for a slice knot K, if 0–surgery is performed on K, the resulting 3–manifold $M(K,0)$ bounds a homology $S^1 \times B^3$, W. This W is constructed by removing a tubular neighborhood of a slice disk for K in the 4–ball. The existence of the tubular neighborhood is equivalent to the existence of the normal bundle.

In a different direction, Freedman's theorem implied that in the topological locally flat category all knots of Alexander polynomial one are slice. To understand why this is a consequence, note first the following.

THEOREM 10.1.1. *For a knot K, if $M(K,0)$ bounds a homology $S^1 \times B^3$, W, with $\pi_1(W) = \mathbb{Z}$ then K is slice.*

Proof We have that $M(K,0)$ is formed from S^3 by removing a solid torus and replacing it with another solid torus. Performing 0–surgery on the core, C, of that solid torus returns S^3. Attach a 2–handle to W with framing 0 to C. The resulting manifold is a homotopy ball with boundary S^3, and hence, by the Poincaré conjecture, is homeomorphic to B^4. The cocore of that added 2–handle is a slice disk for the boundary of the cocore, which can be seen to be the original K.

Freedman observed that a surgery obstruction to finding such a manifold W is determined by the Seifert form, and for a knot of Alexander polynomial one that is the only obstruction, and it vanishes.

10.1.1. Extensions. Is it possible that more delicate arguments using 4–dimensional surgery might yield stronger results, showing that other easily identified classes of algebraically slice knots are slice, based only on the Seifert form of the knot? The following result indicates that the answer is no.

THEOREM 10.1.2. *If $\Delta_K(t)$ is nontrivial then there are two nonconcordant knots having that Alexander polynomial.*

This result was first proved in [**?**] where there was the added constraint that the Alexander polynomial is not the product of cyclotomic polynomials $\phi_n(t)$ with n divisible by three distinct primes. The condition on Alexander polynomials is technical, assuring that some prime power branched cover is not a homology sphere. Taehee Kim [**?**] has shown this condition is not essential in particular cases, and in unpublished work he has shown that the result applies for all nontrivial Alexander polynomials.

10.2. Smooth knot concordance

In 1983 Donaldson [**?**] discovered new constraints on the intersection forms of smooth 4–manifolds. This and subsequent work soon yielded the following theorem.

THEOREM 10.2.1. *Suppose that X is a smooth closed 4–manifold and $H_1(X, \mathbb{Z}_2) = 0$. If the intersection form on $H_2(X)$ is positive definite then the form is diagonalizable. If the intersection form is even and definite, and hence of the type $nE_8 \oplus mH$, where H is the standard 2–dimensional hyperbolic form, then if $n > 0$, it follows that $m > 2$.*

This result is sufficient to prove that many knots of Alexander polynomial one are not slice. The details of any particular example cannot be presented here, but the connections with Theorem 10.2.1 are easily explained.

Let $M(K, 1)$ denote the 3–manifold constructed as 1–surgery on K. Then $M(K, 1)$ bounds the 4–manifold W constructed by adding a 2–handle to the 4–ball along K with framing 1. If K is slice, the generator of $H_2(W)$ is represented by a 2–sphere with self-intersection number 1. A tubular neighborhood of that sphere can be removed and replaced with a 4–ball, showing that $M(K,1)$ bounds a homology ball, X. If $M(K,1)$ also bounds a 4–manifold Y (say simply connected) with intersection form of the type obstructed by Theorem 10.2.1, then a contradiction is achieved using the union of X and Y.

As an alternative approach, notice that if K is slice, the 2–fold branched cover of S^3 branched over K, M_2, bounds the $\mathbb{Z}_2$–homology ball formed as the 2–fold branched cover of B^4 branched over the slice disk. Hence, if M_2 is known to bound a simply connected 4–manifold with one of the forbidden forms of Theorem 10.2.1, then again a contradiction is achieved.

It seems that prior to Donaldson's work it was known that either of these approaches would be applicable to proving that particular polynomial one knots are not slice, but these arguments were not published. In particular, following the announcement of Donaldson's theorem it immediately was known that the pretzel knot $K(-3,5,7)$ and the untwisted double of the trefoil (Akbulut) are not slice. Early papers presenting details of such arguments include [**?**, **?**] where it was shown that there are topologically slice knots of infinite order in smooth concordance.

10.2.1. Further advances. Continued advances in smooth 4–manifold theory have led to further understanding of the knot slicing problem. In particular, proving that large classes of Alexander polynomial one knots are not slice has fallen to algorithmic procedures. Notable among this work is that of Rudolph [**?**, **?**, **?**]. Here we outline briefly the approach using Thurston-Bennequin numbers, as described by Akbulut and Matveyev in the paper [**?**].

The 4–ball has a natural complex structure. If a 2–handle is added to the 4–ball along a knot K with appropriate framing, which we call f for now, the resulting manifold W will itself be complex. According to [**?**], W will then embed in a closed Kahler manifold X. Further restrictions on the structure of X are known to hold, and with these constraints the adjunction formula of Kronheimer and Mrowka [**?**] applies to show that no essential 2–sphere in X can have self-intersection greater than or equal to -1.

On the other hand, if K were slice and the framing f of K were greater than -2, such a sphere would exist. The appropriate framing f mentioned above depends on the choice of representative of K, not just its isotopy class. If the representative is $\mathbf{K}$, then $f = tb(\mathbf{K}) - 1$, where $tb(\mathbf{K})$ is the Thurston-Bennequin number, easily computed from a diagram for $\mathbf{K}$.

Applying this, both Akbulut-Matveyev [?] and Rudolph [?] have given simple proofs that, for instance, all iterated positive twisted doubles of the right handed trefoil are not slice.

Although these powerful techniques have revealed a far greater complexity to the concordance group than had been expected, as of yet they seem incapable of addressing some of the basic questions: for instance the slice implies ribbon conjecture and problems related to torsion in the concordance group.

10.3. Higher order obstructions and the filtration of $\mathcal{C}_{top}$

Recent work of Cochran, Orr, and Teichner has demonstrated a deep structure to the topological concordance group. This is revealed in a filtration of the concordance group by an infinite sequence of subgroups:

$$\cdots \mathcal{F}_{2.0} \subset \mathcal{F}_{1.5} \subset \mathcal{F}_{1} \subset \mathcal{F}_{.5} \subset \mathcal{F}_{0} \subset \mathcal{C}.$$

This approach has successfully placed known obstructions to the slicing problem—the Arf invariant, algebraic sliceness, and Casson-Gordon invariants—as the first in an infinite sequence of invariants. Of special significance is that each level of the induced filtration of the concordance group has both an algebraic interpretation and a geometric one. Here we can offer a simplified view of the motivations and consequences of their work, and in that interest will focus on the $\mathcal{F}_n$ with n a nonnegative integer.

To begin, suppose that $M(K,0)$, 0–surgery on a knot K, bounds a 4–manifold W with the homology type and intersection form of $S^1 \times B^3 \#_n S^2 \times S^2$. Such a W will exist if and only if the Arf invariant of K is trivial. Constructing one such W is fairly simple in this case. Push a Seifert surface F for K into B^4 and perform surgery on B^4 along a set of curves on F representing a basis of a metabolizer for its intersection form, with the additional condition that it represents a metabolizer for the $\mathbb{Z}_2$–Seifert form. (Finding such a basis is where the Arf invariant condition appears.) When performing the surgery, the surface F can be ambiently surgered to become a disk, and the complement of that disk is the desired W.

If a generating set of a metabolizer for the intersection form on $H_2(W)$ could be represented by disjoint embedded 2–spheres, then surgery could be performed on W to convert it into a homology $S^1 \times B^3$. It would quickly follow that K would be slice in a homology 4–ball bounded by S^3.

In the higher dimensional analog (of the concordance group of knotted $(2k-1)$–spheres in S^{2k+1}, $k>1$), there is an obstruction (to finding this family of spheres) related to the twisted intersection form on $H_{k+1}(W, \mathbb{Z}[\pi_1(W)])$, or, equivalently, related to the intersection form on the universal cover of W. In short, the intersection form of W should have a metabolizer that lifts to a metabolizer in the universal cover of W. In this higher dimensional setting, if the obstruction vanishes then, via the Whitney trick, the metabolizer for W can be realized by embedded spheres and W can be surgered as desired. This viewpoint on knot concordance has its roots in the work of Cappell and Shaneson [**?**].

Whether in high dimensions or in the classical setting, the explicit construction of a W described earlier in this section yields a W with cyclic fundamental group. This obstruction is thus determined solely by the infinite cyclic cover and vanishes for algebraically slice knots. Of course, in higher dimensions algebraically slice knots are slice. Clearly something more is needed in the classical case.

In light of the Casson-Freedman approach to 4–dimensional surgery theory, in addition to finding immersed spheres representing a metabolizer for W, one needs to find appropriate dual spheres in order to convert the immersed spheres into embeddings. The Cochran-Orr-Teichner filtration can be interpreted as a sequence of obstructions to finding a family of spheres and dual spheres. To describe the filtration, we denote $\pi^{(0)} = \pi = \pi_1(W)$ and let $\pi^{(n)}$ be the derived subgroup: $\pi^{(n+1)} = [\pi^{(n)}, \pi^{(n)}]$.

Definition 10.3.1. *A knot K is called n–solvable if there exists a (spin) 4–manifold W with boundary $M(K,0)$ such that: (a) the inclusion map $H_1(M(K,0)) \to H_1(W)$ is an isomorphism; (b) the intersection form on $H_2(W, \mathbb{Z}[\pi/\pi^{(n)}])$ has a dual pair of self-annihilating submodules (with respect to intersections and self-intersections), L_1 and L_2; and (c) the images of L_1 and L_2 in $H_2(W)$ generate $H_2(W)$.*

(Here and in what follows we leave the description of $n.5$–solvability to [**?**].)

There are the following basic corollaries of the work in [**?**].

Theorem 10.3.2. *If the Arf invariant of a knot K is 0, then K is 0–solvable. If K is 1–solvable, K is algebraically slice. If K is 2–solvable, Casson–Gordon type obstructions to K being slice vanish. If K is slice, K is n–solvable for all n.*

One of the beautiful aspects of [**?**] is that this very algebraic formulation is closely related to the underlying topology. For those familiar with the language of Whitney towers and gropes, we have the following theorem from [**?**].

THEOREM 10.3.3. *If K bounds either a Whitney tower or a grope of height $n+2$ in B^4, then K is n–solvable.*

Define $\mathcal{F}_n$ to be the subgroup of the concordance group consisting of n–solvable knots. One has the filtration (where we have dropped the $n.5$–subgroups).

$$\cdots \mathcal{F}_3 \subset \mathcal{F}_2 \subset \mathcal{F}_1 \subset \mathcal{F}_0 \subset \mathcal{C}.$$

Beginning with [?] and culminating in [?], there is the following result.

THEOREM 10.3.4. *For all n, the quotient group $\mathcal{F}_n/\mathcal{F}_{n+1}$ is infinite and $\mathcal{F}_2/\mathcal{F}_3$ is infinitely generated.*

Describing the invariants that provide obstructions to a knot being in $\mathcal{F}_n$ is beyond the scope of this survey. However, two important aspects should be mentioned. First, [?] identifies a connection between n-solvability and the structure and existence of metabolizers for linking forms on

$$H_1(M(K,0), \mathbb{Z}[\pi_1(M(K,0))/\pi_1(M(K,0))^{(k)}]), \quad k \leq n,$$

generalizing the fact that for algebraically slice knots the Blanchfield pairing of the knot vanishes.

The second aspect of proving the nontriviality of $\mathcal{F}_n/\mathcal{F}_{n+1}$ is the appearance of von Neumann signatures for solvable quotients of the knot group. Though difficult to compute in general, [?] demonstrates that if K is built as a satellite knot, then in special cases, as with the Casson-Gordon invariant, the value of this complicated invariant is related to the Tristram-Levine signature function of the companion knot. More precisely, if a knot K is built from another knot by removing an unknot U that lies in $\pi^{(n)}$ of the complement and replacing it with the complement of a knot J, then the change in a particular von Neumann η–invariant of the $\pi^{(n)}$–cover is related to the integral of the Tristram-Levine signature function of J, taken over the entire circle. The Cheeger-Gromov estimate for these η–invariants can then be applied to show the nonvanishing of the invariant by choosing J in a way that the latter integral exceeds the estimate. This construction generalizes in a number of ways the one used in applications of the Casson-Gordon invariant described earlier, which applied only in the case that $U \in \pi^{(1)}$ and $U \notin \pi^{(2)}$. Furthermore, the Casson-Gordon invariant is based on a finite dimensional representation where here the representation becomes infinite dimensional. In the construction of [?] it is also required that one work with a family of unknots; a single curve U will not suffice.